Book of Abstracts

for the

The 24th Conference of the International Linear Algebra Society

School of Mathematical and Statistical Sciences
National University of Ireland, Galway

20–24 June, 2022

Contents

Proceedings

Linear Algebra and its Applications is pleased to announce a special issue on the occasion of the 24th Conference of the International Linear Algebra Society (ILAS) at the National University of Ireland, Galway, June 20-24, 2022. Papers corresponding to talks given at the conference should be submitted by December 1, 2022 via the Elsevier Editorial System.

Special editors for ILAS 2022 issue are:

- Nicolas Gillis
- Rachel Quinlan
- Clément de Seguins Pazzis
- Helena Šmigoc

Peter Šemrl is the responsible Editor-in-Chief of LAA for this special issue.

Schedules

Schedule Overview: mornings

Time	Monday, June 20	Tuesday, June 21	Wednesday, June 22	Thursday, June 23	Friday, June 24	
9:00	Registration and coffee (from 8:30) Fáilte! Opening remarks (09:50)	**Plenary 3** Nicolas Gillis *O'Flaherty Theatre* Historical tour on the nonnegative rank	**Plenary 5** Clément de Seguins Pazzis *O'Flaherty Theatre* Decomposing matrices into quadratic ones **Conference photo**	**Plenary 7** Misha Kilmer (SIAG/LA Lecture) *O'Flaherty Theatre* Bridging the divide: from matrix to tensor algebra for optimal approximation and compression	**Plenary 9** Shmuel Friedland (LAMA Lecture) *O'Flaherty Theatre* Rank of a tensor and quantum entanglement	9:00
9:30						9:30
10:00	**Plenary 1** Paul van Dooren (Israel Gohberg ILAS-IWOTA Lecture) *O'Flaherty Theatre* Strongly minimal self-conjugate linearizations for polynomial and rational matrices	Coffee	Coffee	Coffee & Poster session	Coffee	10:00
10:30		**Parallel Session 3** **(120 mins - 4 talks)** M2: Nonnegative matrices (AC214) M4: Quantum Information (Anderson) M5: Comb. matrix theory (AC201) M7: General preservers (AC215) M8: Distance matrices of graphs (AC202) M9: Linear algebra education (O' Flaherty) M22: Coding theory (AC213) Contributed 3A and M10 (AC203) Contributed 3B (AC204)	**Parallel Session 5** (90 mins - 3 talks) M4: Quantum information (Anderson) M6: Inv. Eig. Prob. for graphs (AC213) M7: General preservers (AC215) M10: Numerical linear algebra for PDEs (AC201) M15: Companion matrix forms (D'Arcy Thompson) M16: Riordan arrays (AC214) M18: Kemeny's constant (O'Flaherty) M20: Special matrices (AC203) Contributed 5A (AC204) Contributed 5B (AC202)	**Parallel Session 6** (120 mins - 4 talks) M8: Distance matrices of graphs (AC202) M10: Numerical linear algebra for PDEs (AC201) M11: Legacy of Richard Brualdi (O' Flaherty) M15: Companion matrix forms (D'Arcy Thompson) M18: Kemeny's constant (AC215) M21: Tensors (AC214) M22: Coding Theory (AC213) Contributed 6A (AC204) Contributed 6B (Anderson)	**Parallel Session 8** (120 mins - 4 talks) M4: Quantum information (Anderson) M5: Comb. matrix theory (AC201) M9: Linear algebra education (O' Flaherty) M15: Companion matrix forms (D'Arcy Thompson) M16: Riordan arrays (AC214) M17: Designs and codes (AC202)	10:30
11:00	**Parallel Session 1** (120 mins - 4 talks) M3: Copositive matrices (D'Arcy Thompson) M4: Quantum information (Anderson) M5: Comb. matrix theory (AC201) M7: General preservers (AC215) M13: Rigidity & matrix completion (AC204) M16: Riordan arrays (AC214) M17: Designs and codes (AC202) Contributed 1 (AC203)					11:00
11:30						11:30
12:00			**Plenary 6** Cristiane Tretter *O'Flaherty Theatre* From finite to infinite dimensions: Chances and challenges in spectral theory			12:00
12:30		Lunch		Lunch	**Plenary 10** Vilmar Trevisan *O'Flaherty Theatre* Eigenvalue Location of Symmetric Matrices	12:30
13:00	Lunch		Excursions			13:00

Schedule Overview: afternoons

Time	Monday, June 20	Tuesday, June 21	Wednesday, June 22	Thursday, June 23	Friday, June 24
			Excursions		Slán go fóill! End of conference
14:00		**Parallel Session 4** (120 mins - 4 talks) M1: Graph spectra (AC202) M3: Copositive matrices (D'Arcy Thompson) M11: Legacy of Richard Brualdi (O' Flaherty) M12: Matrix positivity (AC201) M14: History of linear algebra (Anderson) M21: Tensors for signals & systems (AC215) M22: Coding Theory (AC213) Contributed 4A (AC203) Contributed 4B (AC204)		**Parallel Session 7** (90 mins - 3 talks) M3: Copositive matrices (D'Arcy Thompson) M6: Inv. Eig. Prob. for graphs (AC213) M10: Numerical lin. alg. for PDEs (AC201) M13: Rigidity & matrix completions (AC204) M15: Companion matrix forms (Anderson) M16: Riordan arrays (AC214) M17: Designs and codes (AC202) M18: Kemeny's constant (O'Flaherty) MS20: Special matrices (AC203) Contributed 7 (AC215)	
14:30	**Parallel Session 2** **(120 mins - 4 talks)** M1: Graph spectra (D'Arcy Thompson) M2: Nonnegative matrices (Anderson) M6: Inv. Eig. Prob. for graphs (AC213) M9: Linear algebra education (O'Flaherty) M12: Matrix positivity (AC201) M20: Special matrices (AC203) M21: Tensors (AC215) Contributed 2A (AC214) Contributed 2B (AC202)				
15:00					
15:30				Coffee & Poster session	
16:00		Coffee		**Plenary 8** Monique Laurent *O'Flaherty Theatre* Graphs, copositive matrices, and sums of squares of polynomials	
16:30	Coffee	**Plenary 4** Patrick Farrell *O'Flaherty Theatre* Reynolds-robust preconditioners for the stationary incompressible Navier-Stokes and MHD equations			
17:00	**Plenary 2** Pauline van den Driessche (Hans Schneider Prize Lecture) *O'Flaherty Theatre* Linear Algebra is Everywhere: a Duo of Examples from Mathematical Biology			ILAS Business Meeting *O'Flaherty Theatre*	
17:30		ILAS Board Meeting			

Monday, 20 June, Morning

10:00	O'Flaherty Theatre	Plenary Sessions	**Paul Van Dooren**	p21
Strongly minimal self-conjugate linearizations for polynomial and rational matrices				
11:00	AC203	Contrib. 1	**James R. Weaver**	p249
Blocked Triangular Patterns and their Symmetry Groups				
11:30	AC203	Contrib. 1	**Richard Hollister**	p250
Majorization and Triangular Polynomial Matrices				
12:00	AC203	Contrib. 1	**D. Steven Mackey**	p251
Spectral Localization in Polynomial and Rational Matrices				
12:30	AC203	Contrib. 1	**Edward Poon**	p252
Circular higher rank numerical range and factorization of singular matrix polynomials				
11:00	D'Arcy Thompson	MS-3	**Damjana Kokol Bukovšek**	p47
Completely positive factorizations associated with Euclidean distance matrices corresponding to an ...				
11:30	D'Arcy Thompson	MS-3	**Helena Šmigoc**	p48
Symmetric Nonnegative Trifactorization Rank				
12:00	D'Arcy Thompson	MS-3	**Qinghong Zhang**	p49
The Maximal Angle between 5×5 Positive Semidefinite and 5×5 Non-negative matrices				
12:30	D'Arcy Thompson	MS-3	**Mirjam Dür**	p50
Factorization of Completely Positive Matrices				
11:00	Anderson	MS-4	**Julio de Vicente**	p58
Asymptotic survival of genuine multipartite entanglement in noisy quantum networks depends on the t...				
11:30	Anderson	MS-4	**Alexander Müller-Hermes**	p59
Entanglement annihilation between cones				
12:00	Anderson	MS-4	**Sander Gribling**	p60
Mutually unbiased bases: polynomial optimization and symmetry				
12:30	Anderson	MS-4	**Mizanur Rahaman**	p61
An Extension of Bravyi-Smolin's Construction for UMEBs				
11:00	AC201	MS-5	**Michael Tait**	p71
Two conjectures on the spread of graphs				
11:30	AC201	MS-5	**Mark Kempton**	p72
Algebraic Connectivity and the Laplacian Spread				
12:00	AC201	MS-5	**Sebastian M. Cioabă**	p73
Spectral Moore Theorems for Graphs and Hypergraphs				
12:30	AC201	MS-5	**Xiaohong Zhang**	p74
Oriented Cayley graphs with all eigenvalues being rational multiples of each other				
11:00	AC215	MS-7	**Antonio M. Peralta**	p93
Distance-preserving bijections between sets of invertible elements in unital Jordan-Banach algebras				
11:30	AC215	MS-7	**Tamás Titkos**	p94
On isometric rigidity of Wasserstein spaces				
12:00	AC215	MS-7	**Jerónimo Alaminos**	p95
On property ($\mathbb{B}$) and zero product determined Banach algebras				

11:00	AC204	MS-13	**Derek Kitson**	p152
Graph rigidity in cylindrical spaces				
11:30	AC204	MS-13	**Signe Lundqvist**	p153
When is a rod configuration infinitesimally rigid?				
12:00	AC204	MS-13	**John Hewetson**	p154
Global Rigidity of Frameworks in Non-Euclidean Normed Planes				

11:00	AC214	MS-16	**Minho Song**	p179
Enumerative results for connected bipartite non-crossing geometric graphs				
11:30	AC214	MS-16	**Bumtle Kang**	p180
On claw-free Toeplitz graphs				
12:00	AC214	MS-16	**Naiomi T. Cameron**	p181
A Riordan Array Approach to Some Problems involving Lattice Paths, Trees and Partitions				

11:00	AC202	MS-17	**Santiago Barrera Acevedo**	p190
Cocyclic Two-Circulant Core Hadamard Matrices				
11:30	AC202	MS-17	**Andrea Švob**	p191
On some constructions of divisible design Cayley graphs and digraphs				
12:00	AC202	MS-17	**Guillermo Nuñez Ponasso**	p192
The Maximal Determinant Problem and Generalisations				
12:30	AC202	MS-17	**Ian Wanless**	p193
Perfect 1-factorisations and Hamiltonian Latin squares				

Monday, 20 June, Afternoon

14:30	AC214	Contrib. 2A	**Marina Arav**	p254
A characterization of signed graphs with stable maximum nullity at most two				
15:00	AC214	Contrib. 2A	**Hein van der Holst**	p256
A topological characterization of signed graphs with stable positive semidefinite maximum nullity a...				
15:30	AC214	Contrib. 2A	**Milica Anđelić**	p257
Inverse of a signless Laplacian matrix of a non-bipartite graph				
16:00	AC214	Contrib. 2A	**Vicenç Torra**	p258
Graph addition: properties for its use for graph protection				
14:30	AC202	Contrib. 2B	**Frank Uhlig**	p253
New Connections between Static Matrices A, Zhang Neural Networks, and Parameter-Varying Matrix Fl...				
15:00	AC202	Contrib. 2B	**Tom Asaki**	p255
Null-Space Projects for Intermediate Students: Tomography, Cryptography, and More				
14:30	D'Arcy Thompson	MS-1	**Margarida Mitjana**	p32
PageRank: a different point of view				
15:00	D'Arcy Thompson	MS-1	**Suil O**	p33
Eigenvalues, spanning trees, and connected parity factors in regular graphs				
15:30	D'Arcy Thompson	MS-1	**Luiz Emilio Allem**	p34
Randić Energy and Index				
16:00	D'Arcy Thompson	MS-1	**James Borg**	p35
Graphs Reconstructible from One Card and a One–Dimensional Eigenspace				
14:30	Anderson	MS-2	**Miriam Pisonero**	p40
Universal Realizability in Dimension 5 with Trace Zero: nonreal case				
15:00	Anderson	MS-2	**Carlos Marijuán**	p41
Universal Realizability in Dimension 5 with Trace Zero: real case				
15:30	Anderson	MS-2	**Robert Perry, Jonathan Ta**	p42
Kronecker Products of Perron Similarities				
14:30	AC213	MS-6	**Shaun Fallat**	p83
On the maximum multiplicity of the kth largest eigenvalue of a graph.				
15:00	AC213	MS-6	**Franklin Kenter**	p84
A zero forcing menagerie: the ordered multiplicity inverse eigenvalue sequence problem, powers of g...				
15:30	AC213	MS-6	**Mary Flagg**	p85
The Strong Nullity Interlacing Property				
16:00	AC213	MS-6	**Bryan Curtis**	p86
Strong Spectral Norm Property				
14:30	O'Flaherty	MS-9	**Anthony Cronin and Sepideh Stewart**	p110
Analysis of Tutors' Feedback After Responding to Linear Algebra Students' Queries				
15:00	O'Flaherty	MS-9	**Ann Sophie Stuhlmann**	p111
Interactionist perspective on negotiation processes of students' different understandings during s...				
15:30	O'Flaherty	MS-9	**Michelle Zandieh**	p112
Linear combinations of vectors in Inquiry-Oriented Linear Algebra (IOLA)				
16:00	O'Flaherty	MS-9	**John Sheekey**	p113
Incorporating Tensors into Linear Algebra Courses				

14:30	AC201	MS-12	**Paul Barry**	p144
Riordan arrays: structure and positivity				
15:00	AC201	MS-12	**Prateek Kumar Vishwakarma**	p145
Positivity preservers forbidden to operate on diagonal blocks				
15:30	AC201	MS-12	**Daniel Carter**	p146
An Atomic Viewpoint of the Totally Positive Completion Problem				
16:00	AC201	MS-12	**Mika Mattila**	p147
Maximizing the number of positive eigenvalues of an LCM matrix				

14:30	AC203	MS-20	**Susana Furtado**	p210
Efficient vectors for perturbed consistent matrices				
15:00	AC203	MS-20	**Richard Ellard**	p211
An algorithmic approach to the Symmetric Nonnegative Inverse Eigenvalue Problem				
15:30	AC203	MS-20	**Sirani M. Perera**	p212
A Low-complexity Algorithm to Uncouple the Mutual Coupling Effect in Antenna Arrays				
16:00	AC203	MS-20	**Natália Bebiano**	p213
The periodic pseudo-Jacobi inverse eigenvalue problem				

14:30	AC215	MS-21	**Borbala Hunyadi**	p220
Structured Tensor Decompositions in Functional Neuroimaging: Estimating the Hemodynamic Response				
15:00	AC215	MS-21	**Vicente Zarzoso**	p221
Tensor Decomposition of ECG Records for Persistent Atrial Fibrillation Analysis				
15:30	AC215	MS-21	**Orly Alter**	p222
Multi-Tensor Decompositions for Personalized Cancer Medicine				
16:00	AC215	MS-21	**Nico Vervliet**	p223
A quadratically convergent proximal algorithm for nonnegative tensor decomposition				

17:00	O'Flaherty Theatre	Plenary Sessions	**Pauline van den Driessche**	p22
Linear Algebra is Everywhere: a Duo of Examples from Mathematical Biology				

Tuesday, 21 June, Morning

09:00	O'Flaherty Theatre	Plenary Sessions	**Nicolas Gillis**	p23
Historical tour on the nonnegative rank				
10:30	AC203	Contrib. 3A	**Ivana Šain Glibić**	p260
Importance of the deflation process for the solution of quartic eigenvalue problem				
11:00	AC203	Contrib. 3A	**Avleen Kaur**	p261
How the Friedrichs angle leads to lower bounds on the minimum singular value				
11:30	AC203	Contrib. 3A	**George Hutchinson**	p263
On the enumeration and properties of complex matrix scalings				
10:30	AC204	Contrib. 3B	**Dmitry Savostyanov**	p259
Tensor product approach to epidemiological models on networks				
11:00	AC204	Contrib. 3B	**Ryan Wood**	p262
Dynamic Katz and Related Network Measures				
11:30	AC204	Contrib. 3B	**Cheolwon Heo**	p264
The Complexity of the Matroid-homomorphism problems				
12:00	AC204	Contrib. 3B	**Sophia Keip**	p265
Kirchberger's Theorem for Complexes of Oriented Matroids				
10:30	AC214	MS-2	**Rapahel Loewy**	p43
On polynomials preserving nonnegative matrices				
11:00	AC214	MS-2	**A.M. Encinas**	p44
Bisymmetric Nonnegative Jacobi Matrix Realizations				
11:30	AC214	MS-2	**Julio Moro**	p45
A combinatorial characterization of lists realizable by compensation in the SNIEP				
10:30	Anderson	MS-4	**Chi-Kwong Li**	p62
Some results and problems in Quantum Tomography				
11:00	Anderson	MS-4	**Claus Koestler**	p63
Central limit theorems for braided coin tosses				
11:30	Anderson	MS-4	**Darian Mclaren**	p64
Evaluating Quantum Instruments				
10:30	AC201	MS-5	~~**Rachel Quinlan**~~	p75
~~Alternating sign matrices of finite multiplicative order~~				
11:00	AC201	MS-5	**Jephian C.-H. Lin**	p76
Comparability and cocomparability bigraphs				
11:30	AC201	MS-5	**Gary Greaves**	p77
Spectral restrictions for certain symmetric ± 1-matrices with applications to equiangular lines				
12:00	AC201	MS-5	**M.J. de la Puente**	p78
Orthogonality for $(0, -1)$ tropical normal matrices				
10:30	AC215	MS-7	**Peter Šemrl**	p96
Automorphisms of effect algebras				
11:00	AC215	MS-7	**Mark Pankov**	p97
Adjacency preserving transformations of conjugacy classes of finite-rank self-adjoint operators				
11:30	AC215	MS-7	**Janko Bračič**	p98
Collineations of a linear transformation				

10:30	AC202	MS-8	**Aida Abiad**	p103
Extending a conjecture of Graham and Lovász on the distance characteristic polynomial				
11:00	AC202	MS-8	**Projesh Nath Choudhury**	p104
Blowup-polynomials of graphs				
11:30	AC202	MS-8	**Carlos A. Alfaro**	p105
Distance ideals of graphs				
12:00	AC202	MS-8	~~**Lorenzo Ciardo**~~	p106
~~Two moments for trees~~				

10:30	O'Flaherty	MS-9	**Sepideh Stewart, Anthony Cronin**	p114
Students' Perspectives on Proofs in Linear Algebra: Ways of Thinking and Ways of Understanding in . . .				
11:00	O'Flaherty	MS-9	**Megan Wawro**	p115
Student Reasoning about Linear Algebra in Quantum Mechanics				
11:30	O'Flaherty	MS-9	**Amanda Harsy, Michael Smith**	p116
Application Approach to Teaching Linear Algebra				
12:00	O'Flaherty	MS-9	**Frank Uhlig**	p117
16 Questions and Answers for a Modern first Linear Algebra and Matrix Theory Course				

12:00	AC203	MS-10	**Niall Madden**	p123
A boundary-layer preconditioner for singularly perturbed convection diffusion problems				

10:30	AC213	MS-22	**Geertrui Van de Voorde**	p233
The dual code of points and lines in a projective plane				
11:30	AC213	MS-22	**Ignacio F. Rúa**	p235
Codes over finite fields and Galois ring valued quadratic forms				
12:00	AC213	MS-22	**Gary McGuire**	p236
Linearized Polynomials and Galois Groups				

Tuesday, 21 June, Afternoon

14:00	AC203	Contrib. 4A	**Plamen Koev**	p266
Accurate Bidiagonal Decompositions of Structured Totally Nonnegative Matrices with Repeated Nodes				
14:30	AC203	Contrib. 4A	**Michael Tsatsomeros**	p269
The Fiber of P-matrices: the Recursive Construction of All Matrices with Positive Principal Minors				
15:00	AC203	Contrib. 4A	**Raquel Viaña**	p270
Accurate computation of the inverse of Totally Positive collocation matrices of the Lupaş-type (...				
15:30	AC203	Contrib. 4A	**Adi Niv**	p272
Tropical Matrix Identities				
14:00	AC204	Contrib. 4B	**Lauri Nyman**	p267
Perturbation theory of transfer function matrices				
14:30	AC204	Contrib. 4B	~~**Patricia Antunes**~~	p268
~~Bi-additive Models: different types of distributions~~				
15:00	AC204	Contrib. 4B	**Juyoung Jeong**	p271
Weak majorization inequalities in Euclidean Jordan algebras				
15:30	AC204	Contrib. 4B	**Luis Felipe Prieto-Martínez**	p273
Geometric continuity, Riordan matrices and applications				
14:00	AC202	MS-1	**Francesco Belardo**	p36
Identifying the graphs whose (Laplacian) spectral radius is small				
14:30	AC202	MS-1	**Cristina Dalfo**	p37
Almost Moore and largest mixed graphs of diameter two and three				
15:00	AC202	MS-1	**Ivan Damnjanović**	p38
Assigned rational functions of a rooted tree				
14:00	D'Arcy Thompson	MS-3	**Roland Hildebrand**	p51
On the algebraic structure of the copositive cone				
14:30	D'Arcy Thompson	MS-3	**Maxim Manainen**	p52
Generating extreme copositive matrices near matrices obtained from COP-irreducible graphs				
15:00	D'Arcy Thompson	MS-3	**Jordi Tura**	p53
Entangled symmetric quantum states and copositive matrices				
15:30	D'Arcy Thompson	MS-3	**Oliver Mason**	p54
Copositivity and the Riccati Equation				
14:00	O'Flaherty	MS-11	**Geir Dahl**	p135
Richard, Matrices and Polyhedra				
14:30	O'Flaherty	MS-11	**Seth A. Meyer**	p136
Loopy 2-graphs				
15:00	O'Flaherty	MS-11	**K.-T. Howell, N. A. Neudauer**	p137
On the independence of near-vector spaces and their matroids				
15:30	O'Flaherty	MS-11	**Gi-Sang Cheon**	p138
Richard's mathematical legacy that influenced Korea				
14:00	AC201	MS-12	**Hugo J. Woerdeman**	p148
Completing an Operator Matrix and the Free Joint Numerical Radius				
14:30	AC201	MS-12	**Tomack Gilmore**	p149
Coefficientwise total positivity of some matrices defined by linear recurrences				
15:00	AC201	MS-12	**Miklós Pálfia**	p150
Free functions preserving certain partial orders of operators				

14:00	Anderson	MS-14	~~**Rachel Quinlan**~~	p159
~~The invention of character theory (via linear algebra)~~				
14:30	Anderson	MS-14	~~**Zdeněk Strakoš**~~	p160
~~Seventieth anniversary of the conjugate gradient method and what do old papers reveal about our pre...~~				
15:00	Anderson	MS-14	~~**Claude Brezinski**~~	p161
~~The life and the work of André Louis Cholesky~~				
15:30	Anderson	MS-14	~~**Michela Redivo-Zaglia**~~	p162
~~P. Stein and R.L. Rosenberg~~				

14:00	AC215	MS-21	**Isabell Lehmann**	p224
Multi-task fMRI data fusion using Independent Vector Analysis and the PARAFAC2 tensor decomposition				
14:30	AC215	MS-21	**Christos Chatzichristos**	p225
Early soft and flexible fusion of EEG and fMRI via tensor decompositions for multi-subject group an...				
15:00	AC215	MS-21	**Mariya Ishteva**	p226
Parameter Estimation of Parallel Wiener-Hammerstein Systems by Decoupling their Volterra Representa...				
15:30	AC215	MS-21	**Eric Evert**	p227
Existence of best low rank approximations of positive definite tensors				

14:00	AC213	MS-22	**Jean-Guillaume Dumas**	p237
Dynamic Proofs of Retrievability and Verified Evaluation of Secret Dotproducts and Polynomials				
14:30	AC213	MS-22	**Altan Berdan Kılıç**	p238
One-Shot Capacity of Networks with Restricted Adversaries				
15:00	AC213	MS-22	**Jan De Beule**	p239
On Cameron-Liebler sets in projective spaces, and low degree Boolean functions				
15:30	AC213	MS-22	~~**Anurag Bishnoi**~~	p240
~~Trifferent codes and affine blocking sets~~				

16:30	O'Flaherty Theatre	Plenary Sessions	**Patrick E. Farrell**	p24
Reynolds-robust preconditioners for the stationary incompressible Navier–Stokes and MHD equations				

Wednesday, 22 June, Morning

09:00	O'Flaherty Theatre	Plenary Sessions	**Clément de Seguins Pazzis**	p25
Decomposing matrices into quadratic ones				
10:30	AC204	Contrib. 5A	**Milan Hladík**	p274
Strong solvability of restricted interval systems and its applications in quadratic and geometric p...				
11:00	AC204	Contrib. 5A	**Černý Martin**	p276
Monge-like properties in the interval setting				
11:30	AC204	Contrib. 5A	**Matyáš Lorenc**	p278
Interval B-matrices, doubly B-matrices and B_π^R-matrices				
10:30	AC202	Contrib. 5B	**Niel Van Buggenhout**	p275
$\star$-Lanczos procedure for non-autonomous ODEs				
11:00	AC202	Contrib. 5B	**Paula Kimmerling**	p277
Average Mixing Matrices on Dutch Windmill Graphs				
11:30	AC202	Contrib. 5B	**Paola Boito**	p279
Hub and authority centrality measures based on continuous-time quantum walks				
10:30	Anderson	MS-4	**Travis B. Russell**	p65
Universal operator systems generated by projections				
11:00	Anderson	MS-4	**Mark Howard**	p66
Quantum Advantage in Information Retrieval				
11:30	Anderson	MS-4	**Michael Mc Gettrick**	p67
Matrices of interest in higher dimensional quantum walks				
10:30	AC213	MS-6	**Shahla Nasserasr**	p87
The Allows Problem for Graphs with Two Distinct Eigenvalues				
11:00	AC213	MS-6	**Polona Oblak**	p88
On the number of distinct eigenvalues of joins of two graphs				
11:30	AC213	MS-6	**Derek Young**	p89
Inverse eigenvalue and related problems for hollow matrices described by graphs				
10:30	AC215	MS-7	**Apoorva Khare**	p99
Preservers of moment sequences				
11:00	AC215	MS-7	**Dániel Virosztek**	p100
Barycenters of Hellinger distances and Kubo-Ando means as barycenters				
11:30	AC215	MS-7	**Lajos Molnár**	p101
Preservers related to the geometric mean and its variants				
10:30	AC201	MS-10	**Patrick E. Farrell**	p124
A scalable and robust vertex-star relaxation for high-order FEM				
11:00	AC201	MS-10	**Siobhán Correnty**	p125
Flexible infinite GMRES for parameterized linear systems				
11:30	AC201	MS-10	**Kirk M. Soodhalter**	p126
Analysis of block GMRES using a *-algebra-based approach				

10:30	D'Arcy Thompson	MS-15	**Javier Perez**	p164	Error bounds for matrix polynomial eigenvectors
11:00	D'Arcy Thompson	MS-15	**Andrii Dmytryshyn**	p165	Recovering a perturbation of a matrix polynomial from a perturbation of its companion matrix
11:30	D'Arcy Thompson	MS-15	**Aaron Melman**	p166	Applications of companion forms to eigenvalue bounds and scalar polynomials
10:30	AC214	MS-16	**Homoon Ryu**	p182	Competition periods and matrix periods of Boolean Toeplitz matrices
11:00	AC214	MS-16	**Tian-Xiao He**	p183	A Recursive Relation Approach to Riordan Arrays
11:30	AC214	MS-16	**Gukwon Kwon**	p184	Riordan posets and associated matrix algebras
10:30	O'Flaherty	MS-18	**Àlvar Martín**	p200	G-inverses for random walks
11:00	O'Flaherty	MS-18	**Federico Poloni**	p201	An edge centrality measure based on the Kemeny constant
11:30	O'Flaherty	MS-18	**María José Jiménez**	p202	Mean first passage time for distance-biregular graphs
10:30	AC203	MS-20	**João R. Cardoso**	p217	Some special matrices arising in computer vision and related optimization problems
10:30	AC203	MS-20	~~**Domingos M. Cardoso**~~	p214	~~Sharp bounds on the least eigenvalue of a graph determined from edge clique partitions~~
11:00	AC203	MS-20	**Christian Berg**	p215	Self-adjoint operators associated with Hankel moment matrices
11:30	AC203	MS-20	**Rute Lemos**	p216	Inequalities for means of matrices
12:00	O'Flaherty Theatre	Plenary Sessions	**Christiane Tretter**	p26	From finite to infinite dimensions: Chances and challenges in spectral theory

Thursday, 23 June, Morning

09:00	O'Flaherty Theatre	Plenary Sessions	**Misha Kilmer**	p27
Bridging the divide: from matrix to tensor algebra for optimal approximation and compression				
10:00		Poster Session	**Blake McGrane-Corrigan**	p297
Diffusive Stability, Common Lyapunov Functions and Leslie Matrices				
10:00		Poster Session	**John Stewart Fabila-Carrasco**	p294
The Cartesian product of graphs and entropy metrics for graph signals.				
10:00		Poster Session	**Paula Kimmerling**	p296
Recursion of Eigenvectors in Dutch Windmill Graphs				
10:00		Poster Session	**Priyanka Joshi**	p295
Powers of Karpelevič Arcs				
10:00		Poster Session	**V A Kandappan**	p299
Hierarchical Off Diagonal Low Rank Matrices (HODLR) for problems in higher dimensions				
10:00		Poster Session	**Victoria Sánchez Muñoz**	p298
The Mathematics behind the quantification of entanglement in Quantum Mechanics				
10:30	AC204	Contrib. 6A	**Riadh ZORGATI**	p281
Projections, L_p Norms and Stochastic Matrices for Ill-Conditioned Linear Systems of Equations				
11:00	AC204	Contrib. 6A	**Philippe Dreesen**	p283
Solving (Overdetermined) Polynomial Equations				
11:30	AC204	Contrib. 6A	**Eric de Sturler**	p285
Efficient Computation of Parametric Reduced Order Models using Randomization				
12:00	AC204	Contrib. 6A	~~**Alicia Roca**~~	p286
~~The change of the Weierstrass structure under one row perturbation~~				
10:30	Anderson	Contrib. 6B	**André Ran**	p280
Rational matrix solutions to $p(X) = A$				
11:00	Anderson	Contrib. 6B	**Héctor Orera**	p282
Bidiagonal decomposition and accurate computations with matrices of q-integers				
11:30	Anderson	Contrib. 6B	**Yinfeng Zhu**	p284
Hurwitz primitivity and synchronizing automata				
12:00	Anderson	Contrib. 6B	**Raf Vandebril**	p287
Construction of a sequence of orthogonal rational functions				
10:30	AC202	MS-8	**Leslie Hogben**	p107
Spectra of Variants of Distance Matrices of Graphs				
11:00	AC202	MS-8	**Carolyn Reinhart**	p108
The distance matrix and its variants for digraphs				
10:30	AC201	MS-10	**John W. Pearson**	p127
Preconditioned iterative methods for multiple saddle-point systems arising from PDE-constrained opt...				
11:00	AC201	MS-10	**Xiao-Chuan Cai**	p128
A recycling preconditioning method for crack propagation problems				
11:30	AC201	MS-10	**Michal Outrata**	p129
Preconditioning the Stage Equations of Implicit Runge Kutta Methods				
12:00	AC201	MS-10	**Daniel B. Szyld**	p130
Provable convergence rate for asynchronous methods via randomize linear algebra				

10:30	O'Flaherty	MS-11	**Michael William Schroeder (#35)**	p139
On the spectrum of graduate research projects with Richard Brualdi				
11:00	O'Flaherty	MS-11	**John Goldwasser**	p140
Permanents of t-triangular $(0,1)$-matrices				
11:30	O'Flaherty	MS-11	**Jennifer J. Quinn**	p141
Determinants:Digraphs::Pfaffians:Matchings				
12:00	O'Flaherty	MS-11	**Richard A. Brualdi**	p142
Pattern-Avoiding Permutation Matrices				
10:30	D'Arcy Thompson	MS-15	**Luca Gemignani**	p167
Comparison Theorems for Splittings of M-matrices in block Hessenberg Form				
11:00	D'Arcy Thompson	MS-15	**Kevin Vander Meulen**	p168
Using the Hessenberg Form of a Sparse Companion Matrix				
11:30	D'Arcy Thompson	MS-15	**Gianna M. Del Corso**	p169
Orthogonal iterations on companion-like pencils				
12:00	D'Arcy Thompson	MS-15	**Robert M. Corless**	p170
Algebraic Companions				
10:30	AC215	MS-18	**Ángeles Carmona**	p203
Schrödinger random walks and mean first passage time generalization				
11:00	AC215	MS-18	**Karel Devriendt**	p204
The resistance magnitude of a graph				
11:30	AC215	MS-18	**Manuel Miranda**	p205
Biased Advection operators on undirected graphs				
12:00	AC215	MS-18	**Steve Kirkland**	p206
Directed forests and the constancy of Kemeny's constant				
10:30	AC214	MS-21	**Kim Batselier**	p228
Tensor-based methods for large-scale inverse problems in machine learning				
11:00	AC214	MS-21	**Gerwald Lichtenberg**	p229
Multilinear Modeling for Control and Diagnosis				
11:30	AC214	MS-21	**Jan Decuyper**	p230
Decoupling multivariate functions using a nonparametric filtered tensor decomposition				
12:00	AC214	MS-21	**Patrick Gelß**	p231
Tensor-based training of neural networks				
10:30	AC213	MS-22	**Heide Gluesing-Luerssen**	p241
Independent Spaces of q-Polymatroids				
11:00	AC213	MS-22	**Giuseppe Cotardo**	p242
Rank-Metric Lattices				
11:30	AC213	MS-22	**Anina Gruica**	p243
MRD Codes and the Average Critical Problem				
12:00	AC213	MS-22	**Ferdinando Zullo**	p244
From linear to non-linear functions over finite fields				

Thursday, 23 June, Afternoon

14:00	AC215	Contrib. 7	**Madelein van Straaten**	p288
H-selfadjoint mth roots of H-selfadjoint matrices over the quaternions				
14:30	AC215	Contrib. 7	**Dawie Janse van Rensburg**	p289
An alternative canonical form for quaternionic H-unitary matrices.				
15:00	AC215	Contrib. 7	**M. Eulàlia Montoro**	p290
The combinatory under isomorphic lattices of hyperinvariant subspaces				
14:00	D'Arcy Thompson	MS-3	**Naomi Shaked-Monderer**	p55
The $\{+,-,0\}$ sign patterns of inverse doubly nonnegative matrices and inver...				
14:30	D'Arcy Thompson	MS-3	**Sachindranath Jayaraman**	p56
Linear preservers of copositive and completely positive matrices				
14:00	AC213	MS-6	**Rupert Levene**	p90
Spectral arbitrariness for trees fails spectacularly, I				
14:30	AC213	MS-6	**H. Tracy Hall**	p91
Spectral arbitrariness for trees fails spectacularly, II				
14:00	AC201	MS-10	**Davide Palitta**	p131
Matrix equation techniques for certain evolutionary partial differential equations				
14:30	AC201	MS-10	**Conor McCoid**	p132
Extrapolation methods as nonlinear Krylov methods				
15:00	AC201	MS-10	**V A Kandappan**	p133
A Domain Decomposition based preconditioner for Discretised Integral equations in two dimensions				
14:00	AC204	MS-13	**James Cruickshank**	p155
Global Rigidity for Line Constrained Frameworks				
14:30	AC204	MS-13	**Shin-ichi Tanigawa**	p156
A Characterization of Graphs of Super Stable Tensegrities				
15:00	AC204	MS-13	**Sean Dewar**	p157
The number of realisations of a minimally rigid graph in various geometries				
14:00	Anderson	MS-15	**Vanni Noferini**	p171
$\mathbb{DL}(P)$, Bézoutians, and the eigenvalue exclusion theorem for singular matrix polynomial...				
14:30	Anderson	MS-15	**María C. Quintana**	p172
Linearizations of rational matrices from general representations				
15:00	Anderson	MS-15	**A. Satyanarayana Reddy**	p173
Primitive Companion Matrices				
14:00	AC202	MS-17	**Ferdinand Ihringer**	p194
The Density of Complementary Subspaces				
14:30	AC202	MS-17	**Eimear Byrne**	p195
q-Polymatroids and Designs over $GF(q)$				
15:00	AC202	MS-17	**Siripong Sirisuk**	p196
Enumeration of some matrices and free linear codes over finite commutative rings				
14:30	O'Flaherty	MS-18	**Jane Breen**	p207
Kemeny's constant for non-backtracking random walks				
15:00	O'Flaherty	MS-18	**Robert E. Kooij**	p208
Kemeny's Constant for Several Families of Graphs and Real-world Networks				
14:00	AC203	MS-20	~~**Mikhail Tyaglov**~~	p218
~~Tridiagonal matrices with two-periodic main diagonal~~				
16:00	O'Flaherty Theatre	Plenary Sessions	**Monique Laurent**	p28
Graphs, copositive matrices, and sums of squares of polynomials				

Friday, 24 June, Morning

09:00	O'Flaherty Theatre	Plenary Sessions	**Shmuel Friedland**	p29
Rank of a tensor and quantum entanglement				
10:30	AC201	MS-4	**Victoria Sánchez Muñoz**	p68
Quantum Information: the Mathematics behind the quantification of quantum entanglement and the dist...				
11:00	AC201	MS-4	**J. Alejandro Chávez-Domínguez**	p69
Isoperimetric inequalities for quantum graphs				
10:30	Anderson	MS-5	**Enide Andrade**	p79
Combinatorial Perron Parameters and Classes of Trees				
11:00	Anderson	MS-5	**Sooyeong Kim**	p80
Kemeny's constant for a chain of connected graphs with respect to a tree				
11:30	Anderson	MS-5	**Minerva Catral**	p81
Minimum number of distinct eigenvalues allowed by a sign pattern				
12:00	Anderson	MS-5	**Rachel Quinlan**	p75
Alternating sign matrices of finite multiplicative order				
10:30	O'Flaherty	MS-9	**Emily J. Evans**	p118
From beginner to expert, increasing linear algebra fluency and comfort with Python labs.				
11:00	O'Flaherty	MS-9	**Heather Moon and Marie Snipes**	p119
Inspiring Linear Algebra Topics Using Image and Data Applications				
11:30	O'Flaherty	MS-9	**Günhan Caglayan**	p120
Pedagogy of linear combination and the levels of thinking about linear combination				
12:00	O'Flaherty	MS-9	**Damjan Kobal**	p121
Matrix zeros of polynomials				
10:30	D'Arcy Thompson	MS-15	**Froilán Dopico**	p174
Linearizations of matrix polynomials via Rosenbrock polynomial system matrices				
11:00	D'Arcy Thompson	MS-15	**Louis Deaett**	p175
Non-sparse companion matrices				
11:30	D'Arcy Thompson	MS-15	**Roberto Canogar**	p176
Non-sparse Companion Hessenberg Matrices				
12:00	D'Arcy Thompson	MS-15	**Fernando De Terán**	p177
Companion pencils for scalar (and matrix) polynomials in the monomial basis				
10:30	AC214	MS-16	**Emanuele Munarini**	p185
Set coverings				
11:00	AC214	MS-16	**Lou Shapiro**	p186
Pseudo-involutions and palindromes in the Riordan group				
11:30	AC214	MS-16	**Ana Luzón**	p187
Commutators in the Riordan group				
12:00	AC214	MS-16	**Nikolaos Pantelidis**	p188
Quasi-involutions of the Riordan group				
10:30	AC202	MS-17	**Dean Crnković**	p197
q-ary strongly regular graphs				
11:00	AC202	MS-17	**Robert Craigen**	p198
Negacyclic weighing matrices				
11:30	AC202	MS-17	**Cian O'Brien**	p292
Weighted Projections of Alternating Sign Matrices and Latin-like Squares				
12:00	AC202	MS-17	**Andrea Švob**	p191
On some constructions of divisible design Cayley graphs and digraphs				
12:30	O'Flaherty Theatre	Plenary Sessions	**Vilmar Trevisan**	p30
Eigenvalue Location of Symmetric Matrices				

Plenary Sessions

Israel Gohberg ILAS-IWOTA Lecture

Strongly minimal self-conjugate linearizations for polynomial and rational matrices

Paul Van Dooren

Université catholique de Louvain

We prove that we can always construct strongly minimal linearizations of an arbitrary rational matrix from its Laurent expansion around the point at infinity, which happens to be the case for polynomial matrices expressed in the monomial basis. If the rational matrix has a particular self-conjugate structure we show how to construct strongly minimal linearizations that preserve it. The structures that are considered are the Hermitian and skew-Hermitian rational matrices with respect to the real line, and the para-Hermitian and para-skew-Hermitian matrices with respect to the imaginary axis. We pay special attention to the construction of strongly minimal linearizations for the particular case of structured polynomial matrices. The proposed constructions lead to efficient numerical algorithms for constructing strongly minimal linearizations. The fact that they are valid for *any* rational matrix is an improvement on any other previous approach for constructing other classes of structure preserving linearizations, which are not valid for any structured rational or polynomial matrix. The use of the recent concept of strongly minimal linearization is the key for getting such generality.

Strongly minimal linearizations are Rosenbrock's polynomial system matrices of the given rational matrix, but with a quadruple of linear polynomial matrices (i.e. pencils) :

$$L(\lambda) := \begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix},$$

where $A(\lambda)$ is regular, and the pencils $\begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix}$ and $\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}$ have no finite or infinite eigenvalues. Strongly minimal linearizations contain the complete information about the zeros, poles and minimal indices of the rational matrix and allow to recover very easily its eigenvectors and minimal bases. Thus, they can be combined with algorithms for the generalized eigenvalue problem for computing the complete spectral information of the rational matrix.

Our results are inspired by the work of Israel Gohberg and his coauthors.

This is joint work with Froilán M. Dopico and María C. Quintana

Hans Schneider Prize Lecture

Linear Algebra is Everywhere: a Duo of Examples from Mathematical Biology

Pauline van den Driessche

University of Victoria, B.C. Canada

Linear algebra is increasingly important in applications to many areas. To illustrate this statement, two problems in mathematical biology are considered. The first concerns target reproduction numbers as threshold parameters. These are defined, their properties investigated, and then applied to the projection matrix of an invasive weed having three life stages, with the aim of controlling the weed. The second concerns the spread of an infectious disease, such as cholera, in a heterogeneous environment modeled as a community network. The impact of varying the network on the basic reproduction number is quantified by using a group inverse, and control strategy investigated.

Historical tour on the nonnegative rank

Nicolas Gillis

University of Mons

The nonnegative rank of a nonnegative matrix is the minimum number of nonnegative rank-one matrices whose sum is equal to that nonnegative matrix. The notion of nonnegative rank appeared in the 70's in the linear algebra community [1]; see [2] for an early survey. Although the nonnegative rank seems at first sight to be a natural extension of the usual rank of a matrix, it leads to many intriguing questions and its properties are rather different than that of the rank. For example, the nonnegative rank is NP-hard to compute in general (Vavasis, 2010), and there exists a class of n-by-n nonnegative matrices whose usual rank is equal to 3 but whose nonnegative rank is at least $\sqrt{2n}$ (Fiorini, Rothvoss and Tiwary, 2012).

The main goal of this talk is twofold. First, we will highlight some key properties and results on the nonnegative rank with an historical flavour. This includes its geometric interpretation, the gap between the rank and the nonnegative rank, computational complexity results, and the uniqueness of nonnegative rank factorizations. Second, we will review applications where the nonnegative rank arises, including analytical chemistry (Wallace, 1960), geoscience and remote sensing (Imbrie and Van Andel, 1963), computational geometry (Silio 1979, Aggarwal et al. 1989), probability (Suppes and Zanotti, 1981), extended formulations in combinatorial optimization (Yannakakis, 1991), and unsupervised data analysis where nonnegative matrix factorization (NMF, that looks for low-rank approximations with nonnegativity constraint on the factors) has been particularly impactful (Lee and Seung, 1999).

This talk is partly based on the book [3]; in particular Chapter 1.4 (Introduction - History) and Chapter 3 (Nonnegative Rank).

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Reynolds-robust preconditioners for the stationary incompressible Navier–Stokes and MHD equations

Patrick E. Farrell

University of Oxford

When approximating PDEs with the finite element method, large sparse linear systems must be solved. The ideal preconditioner yields convergence that is algorithmically optimal and parameter robust, i.e. the number of Krylov iterations required to solve the linear system to a given accuracy does not grow substantially as the mesh or problem parameters are changed.

Achieving this for the stationary Navier–Stokes equations has proven challenging: LU factorisation is Reynolds-robust but scales poorly with degree of freedom count, while Schur complement approximations such as PCD and LSC degrade as the Reynolds number is increased.

Building on the work of Schöberl, Olshanskii, and Benzi, in this talk we present the first preconditioner for the Newton linearisation of the stationary incompressible Navier–Stokes equations in three dimensions that achieves both optimal complexity and Reynolds-robustness. The exact details of the preconditioner varies with discretisation, but the main idea is to combine augmented Lagrangian stabilisation, a custom multigrid prolongation operator involving local solves on coarse cells, and an additive patchwise relaxation on each level that captures the kernel of the divergence operator.

We present 3D simulations with over one billion degrees of freedom with robust performance from Reynolds number 10 to 5000. We also present recent extensions to apply these ideas to build parameter-robust solvers for the stationary incompressible resistive equations of magnetohydrodynamics.

This is joint work with Fabian Laakmann (Oxford) and Lawrence Mitchell (NVIDIA). Supported by the EPSRC Centre for Doctoral Training in Partial Differential Equations [grant EP/L015811/1], and by EPSRC grants EP/R029423/1 and EP/W026163/1.

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Decomposing matrices into quadratic ones

Clément de Seguins Pazzis

Université de Versailles Saint-Quentin-en-Yvelines

Let $\mathbb{F}$ be an arbitrary field. An element x of an $\mathbb{F}$-algebra is called quadratic when it is annihilated by a polynomial of degree 2 with entries in $\mathbb{F}$. Such elements include the involutions ($x^2 = 1$), the idempotents ($x^2 = x$), the square-zero elements ($x^2 = 0$), quarter turns ($x^2 = -1$) and so on.

Starting from the 1960's, decomposing matrices into quadratic ones has attracted the attention of many researchers, for decomposition into sums as well as decompositions into products. Most notably:

- products of idempotents have been studied by Erdos [3] and Ballantine [1];
- products of two involutions have been characterized by Wonenburger [8], Djoković [2], Hoffmann and Paige [5]; Gustafson et al [4] have proved that every matrix with determinant ± 1 is the product of at most 4 involutions, and no less in general;
- sums of idempotents have been characterized by Wu [10].

This talk will focus on recent breakthroughs in such problems. One of the main ones deals with the so-called "mixed length 2 problem", for which a complete solution has recently been found [6]. In the mixed length 2 problem for sums (respectively, for products), one considers arbitrary fixed polynomials p and q with degree 2 over $\mathbb{F}$, and one asks which square matrices split into $A + B$ (respectively, AB) for matrices A and B such that $p(A) = 0$ and $q(B) = 0$. Many results on the mixed length 2 problem were obtained by J.-H. Wang in the early 1990's, but he stuck to considering matrices over the complex numbers [9], which hides most of the difficulties that arise in the general case.

We will also point to similar decomposition problems in different contexts: stable decompositions (see e.g. [7]), decompositions of endomorphisms of infinite-dimensional vector spaces, decompositions into sums of selfadjoint or skew-selfadjoint endomorphisms, decompositions in orthogonal or symplectic groups.

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LAA Lecture

From finite to infinite dimensions: Chances and challenges in spectral theory

Christiane Tretter

University of Bern, Switzerland

This lecture focuses on chances and challenges in obtaining reliable information on eigenvalues and, more generally, spectra of linear operators. Two aspects will be addressed. First, finite dimensional tools to enclose spectra of infinite dimensional problems will be presented. Spectral bounds in terms of these so-called block numerical ranges [1] improve classical numerical range bounds, both in infinite and finite dimensions. Secondly, infinite dimensional tools to capture spurious eigenvalues of finite dimensional spectral approximations will be showcased. These so-called essential numerical ranges [2], [3], originally designed to enclose essential spectra, turn out to be powerful tools to assess the reliability of finite dimensional spectral approximations for unbounded linear operators. Examples and applications illustrate the abstract results.

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SIAG/LA Lecture

Bridging the divide: from matrix to tensor algebra for optimal approximation and compression

Misha Kilmer

Tufts University

Tensors, also known as multiway arrays, have become ubiquitous as representations for operators or as convenient schemes for storing data. Yet, when it comes to compressing these objects or analyzing the data stored in them, the tendency is to "flatten" or "matricize" the data and employ traditional linear algebraic tools, ignoring higher dimensional correlations/structure that could have been exploited. Impediments to the development of equivalent tensor-based approaches stem from the fact that familiar concepts, such as rank and orthogonal decomposition, have no straightforward analogues and/or lead to intractable computational problems for tensors of order three and higher. In this talk, we will review some of the common tensor decompositions and discuss their theoretical and practical limitations. We then discuss a family of tensor algebras based on a new definition of tensor-tensor products. Unlike other tensor approaches, the framework we derive based around this tensor-tensor product allows us to generalize in a very elegant way all classical algorithms from linear algebra. Furthermore, under our framework, tensors can be decomposed in a natural (e.g. 'matrix-mimetic') way with provable approximation properties and with provable benefits over traditional matrix approximation. In addition to several examples from recent literature illustrating the advantages of our tensor-tensor product framework in practice, we highlight interesting open questions and directions for future research.

Graphs, copositive matrices, and sums of squares of polynomials

Monique Laurent

CWI, Amsterdam, and Tilburg University

This lecture revolves around a central open question, relevant to the computation of the stability number $\alpha(G)$ of a graph $G = ([n], E)$ in discrete optimization, to the cone COP_n of copositive matrices, and to the cone Σ of sums of squares of polynomials. Consider the matrix $M_G = \alpha(G)(I + A_G) - J$, where A_G is the adjacency matrix of G and J is the all-ones matrix, and the associated polynomial $p_G = (x^{\circ 2})^T M_G x^{\circ 2}$ in the squared variables $x^{\circ 2} = (x_1^2, \ldots, x_n^2)$. As is well-known the matrix M_G is copositive and thus the polynomial p_G is globally nonnegative on $\mathbb{R}^n$. The question is whether there exists a *positivity certificate* in the form $(\sum_{i=1}^n x_i^2)^r p_G \in \Sigma$ for some integer $r \in \mathbb{N}$. De Klerk and Pasechnik (2002) conjecture that the answer is positive, in fact already for $r = \alpha(G) - 1$.

Following Parrilo (2000) let $\mathcal{K}_n^{(r)}$ consist of all symmetric matrices M for which the associated polynomial $(\sum_{i=1}^n x_i^2)^r (x^{\circ 2})^T M x^{\circ 2}$ is a sum of squares. These cones form an inner approximation hierarchy of COP_n and they are known to cover its full interior:

$$\mathrm{int}(\mathrm{COP}_n) \subseteq \bigcup_{r \geq 0} \mathcal{K}_n^{(r)} \subseteq \mathrm{COP}_n.$$

The above open question thus asks whether any graph matrix M_G belongs to some cone $\mathcal{K}_n^{(r)}$, a nontrivial question since any M_G lies on the boundary of COP_n. As one of our new results we show that the answer is positive for the class of graphs that do not have any α-critical edge, which corresponds to the case when p_G has finitely many zeros on the unit sphere.

It is known that any 4×4 copositive matrix belongs to $\mathcal{K}_4^{(0)}$, the dual of the cone of doubly-nonnegative matrices. We show that the union of the cones $\mathcal{K}_n^{(r)}$ does not cover COP_n if $n \geq 6$. However it remains open what is the situation for $n = 5$. What we can show is that the Horn matrix M_{C_5} plays a crucial role: it remains only to settle whether any positive diagonal scaling of the Horn matrix belongs to some cone $\mathcal{K}_5^{(r)}$.

We will discuss old and new results around the above questions and related ones, which display a nice interplay between graph structure, optimization, copositive matrices, and real algebraic geometry.

This is based on joint works [1, 2, 3] with Luis Felipe Vargas (CWI, Amsterdam).

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LAMA Lecture

Rank of a tensor and quantum entanglement

Shmuel Friedland

University of Illinois at Chicago

A tensor is a multiarray with $d \geq 3$ indices, which is a vector in the tensor product of d-vector spaces. The rank of a tensor is a minimal number of summands in a decomposition to a sum of rank-one tensors. In this talk we discuss the notions of the generic rank, maximal rank, border rank, symmetric rank and nuclear rank of tensors. We review some known results, open problems, and numerical methods to compute different ranks.

The rank of a tensor is a simple measure of quantum entanglement. A pure quantum state $\mathbf{v}$ of a composite system consisting of d subsystems with n levels each. It is viewed as a vector in the d-fold tensor product of n-dimensional Hilbert space, and can be identified with a tensor with d indices, each running from 1 to n. A quantum state $\mathbf{v}$ is called *entangled* if its not a rank-one tensor: $\mathbf{v} \neq \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_d$, which implies correlations between physical subsystems. A relation between various ranks and norms of a tensor and the entanglement of the corresponding quantum state is revealed.

This is joint work with Wojciech Bruzda and Karol Życzkowski (Jagiellonian University, Krakow).

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Eigenvalue Location of Symmetric Matrices

Vilmar Trevisan

UFRGS-Universidae Federal do Rio Grande do Sul

We address the problem of estimating graph eigenvalues in terms of eigenvalue location, by which we mean determining the number of eigenvalues of a symmetric matrix that lie in any given real interval.

Our algorithms are based on diagonalizing matrices and rely on Sylvester's Law of Inertia. They are either designed for graphs in a particular class, and exploit some special feature of this class, or they rely on a structural decomposition of the input graph.

We show how a simple linear-time tree algorithm can be extended to symmetric matrices whose underlying graph has a tree decomposition of small width. We also describe how a linear-time cograph algorithm can be extended to matrices whose underlying graph has small clique-width.

These algorithms have applications that go beyond estimating eigenvalues of a particular graph, and allow us to obtain properties of an entire class. We illustrate this with applications to the solution of relevant problems in Spectral Graph Theory.

MS-1: Graph spectra

Organisers: Domingos Cardoso, Claudia Justel and Renata del Vecchio

Theme: Spectral graph Theory (SGT) is nowadays a strong research mathematical field relating Linear Algebra with Graph Theory. Many combinatorial properties of graphs can be deduced from the study of the eigenvalues and eigenvectors of matrices that represent them and the converse is also true. On the other hand, its application to problems in Chemistry, Computer Science, Operational Research and Combinatorial Optimization has been intensive with valuable results. Although the SGT beginnings were in Chemistry applications (interpreting the molecular graph eigenvalues), more recently, several new areas, such as quantum physics and communication networks, model their problems by SGT parameters.
This minisymposium will bring together a group of researchers that will present their recent contributions to the area, allowing to establish the general framework of problems addressed in SGT as well as new directions and open problems.

PageRank: a different point of view

Margarida Mitjana

Universitat Politècnica de Catalunya

To compute PageRank in the classical model, it is supposed that for some fixed probability d, a surfer jumps to a random node with probability d (damping factor) and goes to an adjacent node with probability $(1-d)$. In the personalized PageRank, a vector v (teleportation or personalized vector) is also considered. Then, the personalized PageRank is the unique probability eigenvector of the Google matrix associated with the eigenvalue 1. The Google matrix, see [1], is

$$G = (1-d)P + d\,\mathsf{e}\,\mathsf{v},$$

where P is the transition probability matrix and e is the all one vector. Some methods to compute the PageRank consider the M-matrix $\mathsf{I}-\mathsf{G}$, which is singular and weakly diagonally dominant. Other models consider also a constant probability of remaining in the node, the so-called lazy parameter that correspond to consider $\dfrac{\mathsf{I}+\mathsf{P}}{2}$ instead of P, then I_G is a diagonally dominant M-matrix and hence it is nonsingular.

The fundamental centrality measure PageRank implicitly uses Schrödinger operators for its formulation, which corresponds to use diagonally dominant M-matrices. This is due to the presence of the damping parameter for the formulation of the ranking process. Therefore, it is possible, to extend this centrality measure to general Schrödinger operators; that is, to general M-matrices. We plan here to tackle a more realistic model with a wider range of applications. Specifically, we consider in each step of the random walk the importance of both the present state and the state we want to reach. Moreover, the lazy term can be considered as a function instead of a parameter. This model appears when considering a transition probability matrix associated with a symmetric M-matrix (singular or not singular); that is, we can erase the diagonally dominant hypothesis.

This is joint work with Ángeles Carmona, Andrés M.Encinas and M. José Jiménez (Universitat Politècnica de Catalunya). Partially supported by the Departament de Matemàtiques (UPC).

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Eigenvalues, spanning trees, and connected parity factors in regular graphs

Suil O

SUNY-Korea

In this talk, we prove a sharp upper bound for the second largest eigenvalue in an r-regular graph G to guarantee that G contains at least two disjoint spanning trees. By utilizing the result, we prove an upper bound for the second largest eigenvalue in an r-regular graph to guarantee the existence of a connected parity factor.

This is joint work with Donggyu Kim (KAIST) and Zhiwen Wang (Nankai). Supported by the National Research Foundation of Korea, Grant NRF-2020R1F1A1A01048226.

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Randić Energy and Index

Luiz Emilio Allem

Universidade Federal do Rio Grande do Sul

In this talk, we present ongoing work on a conjecture proposed by Gutman, Furtula and Bozkurt [1] about the Randić energy (RE) of graphs. Specifically, they used computational experiments to conjecture that the p-sun and the balanced $(\lceil \frac{n-2}{4} \rceil, \lfloor \frac{n-2}{4} \rfloor)$-double sun are the graphs with largest Randić energy among connected graphs. The p-sun, S^p, is a starlike tree of order $n = 2p + 1$, $p \geq 0$, having p-paths of length 2 and the (p, q)-double sun, $D^{p,q}$, is a tree of order $n = 2(p + q + 1)$, where $p, q \geq 0$, obtained by connecting the centers of a p-sun and a q-sun with an edge.

We show that the family of bipartite graphs with bipartition A, B such that $\deg(b) \leq 2$ for every $b \in B$, called TB - graphs, satisfies the conjecture for n odd, where n is the number of vertices. Next, we extend the results to a more general class of graphs, which we call ATB - graphs. We conclude with some computational experiments about the Randić index R_{-1} of trees.

This includes joint work with Adrián Pastine (UNSL), Gonzalo Molina (UNSL) and Rodrigo O. Braga (UFRGS). Supported by FAPERGS, Grant 21/2551-0002053-9.

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Graphs Reconstructible from One Card and a One–Dimensional Eigenspace

James Borg

University of Malta

The deck D of a graph G is its multiset of one-vertex deleted subgraphs. We prove that a graph G with a given generator of the eigenspace of any simple eigenvalue μ of the 0-1-adjacency matrix is reconstructed uniquely from one μ–card of D, that is, a one-vertex deleted subgraph that does not have μ as an eigenvalue. If the generator of the μ-eigenspace is known to be full, that is if it has no zero entries, the graph is said to be a μ-nut graph. For a μ-nut graph, the reconstruction holds from any card. No two non-isomorphic μ-nut graphs having a common μ–card, have the same associated one-dimensional eigenspace. Moreover two non-isomorphic μ-nut graphs with the same simple eigenvalue and associated eigenspace have no card in common.

This is joint work with Irene Sciriha

Identifying the graphs whose (Laplacian) spectral radius is small

Francesco Belardo

University of Naples Federico II

A connected simple graph has a small (Laplacian) spectral radius if it does not exceed $3/\sqrt{2}$ (resp. $(3/\sqrt{2})^2 = 4.5$). The latter number comes as the limit value for the infinite subdivision of any graph with maximum degree 3 [2]. The identification of (connected) graphs with small spectral radius has been a quite investigated topic in Spectral Graph Theory. In [3] it is proved that for the adjacency spectral radius such graphs have a natural *quipu* structure, that is, the vertices of maximum degree 3 lie either on a path or on a cycle. Recently [1], we have attacked the analogous problem for the signless Laplacian spectral radius. The latter research has relevant consequences for the adjacency and the Laplacian cases. Here, we survey what we got so far.

This is joint work with M. Brunetti (Napoli) and Vilmar Trevisan (Porto Alegre).

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Almost Moore and largest mixed graphs of diameter two and three

Cristina Dalfo

Universitat de Lleida

Almost Moore mixed graphs appear in the context of the *degree/diameter problem* as a class of extremal mixed graphs, in the sense that their order is one unit less than the Moore bound for such graphs. The problem of their existence has been considered just for diameter 2. In this paper, we give a complete characterization of these extremal mixed graphs for diameters 2 and 3. We also derive some optimal constructions for other diameters.

This is joint work with M. A. Fiol (Universitat Politècnica de Catalunya), N. López (Universitat de Lleida), J. M. Miret (Universitat de Lleida).

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Assigned rational functions of a rooted tree

Ivan Damnjanović

Diffine LLC & Faculty of Electronic Engineering, University of Niš

We investigate the spectral properties of rooted trees with the intention of improving the currently existing results that deal with this matter. The concept of an assigned rational function is recursively defined for each vertex of a rooted tree. Afterwards, two mathematical formulas are given which show how the characteristic polynomials of the adjacency and Laplacian matrix can be represented as products of the aforementioned rational functions. In order to demonstrate their general use case scenario, the obtained formulas are subsequently implemented on balanced trees, with a special focus on the Bethe trees. In the end, some of the previously derived results are used in order to construct a tree merging procedure which preserves the spectra of all of the starting trees.

Supported by Diffine LLC.

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MS-2: Spectral properties of non-negative matrices

Organisers: Carlos Marijuán (Universidad de Valladolid) and Pietro Paparella (University of Washington Bothell)

Universal Realizability in Dimension 5 with Trace Zero: nonreal case

Miriam Pisonero

Universidad de Valladolid/IMUVA

The *nonnegative inverse eigenvalue problem* (NIEP) is the problem of finding necessary and sufficient conditions for a list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of complex numbers to be the spectrum of a nonnegative matrix. We say that a realizable list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, of complex numbers, is *universally realizable* if, for every possible Jordan canonical form allowed by Λ, there is a nonnegative matrix with spectrum Λ. The problem of finding necessary and sufficient conditions for a realizable list Λ, of complex numbers, to be universally realizable will be called the *universal realizability problem* (URP). In terms of n, the NIEP is completely solved only for $n \leq 4$, and for $n = 5$ with trace zero. It is clear that for $n \leq 3$ the concepts of universally realizable and realizable are equivalent. The URP is also solved for $n \leq 4$ and the solution is different to the NIEP. In this talk we study the universal realizability of nonreal spectra of size 5 and trace zero and describe a region for the universal realizability of nonreal 5-spectra with trace zero. We use techniques from Graph Theory and from Linear Algebra.

This is a joint work with Ana I. Julio (UCN), C. Marijuán (UVa) and R. L. Soto (UCN). Supported by GIR TAMCO from UVa.

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Universal Realizability in Dimension 5 with Trace Zero: real case

Carlos Marijuán

Universidad de Valladolid/IMUVA

The *nonnegative inverse eigenvalue problem* (NIEP) is the problem of finding necessary and sufficient conditions for a list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of complex numbers to be the spectrum of a nonnegative matrix. We say that a realizable list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, of complex numbers, is *universally realizable* if, for every possible Jordan canonical form allowed by Λ, there is a nonnegative matrix with spectrum Λ. The problem of finding necessary and sufficient conditions for a realizable list Λ, of complex numbers, to be universally realizable will be called the *universal realizability problem* (URP). In terms of n, the NIEP is completely solved only for $n \leq 4$, and for $n = 5$ with trace zero. It is clear that for $n \leq 3$ the concepts of universally realizable and realizable are equivalent. The URP is also solved for $n \leq 4$ and the solution is different to the NIEP. In this talk we characterize the universal realizability of real spectra of size 5 and trace zero. We use techniques from Graph Theory and from Linear Algebra.

This is a joint work with Ana I. Julio (UCN), M. Pisonero (UVa) and R. L. Soto (UCN). Supported by grant PGC2018-096446-B-C21 funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europe.

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Kronecker Products of Perron Similarities

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An invertible matrix is called a Perron similarity if one of its columns and the corresponding row of its inverse are both nonnegative or both nonpositive. Such matrices are of relevance and import in the study of the nonnegative inverse eigenvalue problem. In this talk, Kronecker products of Perron similarities are examined and used to construct ideal Perron similarities all of whose rows are extremal.

This is joint work with Pietro Paparella and Janelle Dockter.

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On polynomials preserving nonnegative matrices

Rapahel Loewy

Technion-Israel Institute of Technology

The Nonnegative Inverse Eigenvalue Problem (NIEP) asks when is a list $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of complex numbers the spectrum of an $n \times n$ nonnegative matrix A. If it is, Λ is said to be realizable and A is a realizing matrix for Λ. This is a well known problem, fully solved only for $n \leq 4$. Making progress on its solution requires to obtain necessary conditions for Λ to be realizable. Motivated by this, Loewy and London defined the following set. Given a positive integer n, let

$$\mathcal{P}_n = \{p \in \mathbb{C}[x] : p(A) \geq 0, \text{ for all } A \geq 0, A \in \mathbb{R}^{n,n}\}.$$

Indeed, if Λ is realizable, so must be the list $p(\Lambda) := (p(\lambda_1), p(\lambda_2), \ldots, p(\lambda_n))$, for any $p \in \mathcal{P}_n$.

It is clear that, for a polynomial p to be in $\mathcal{P}_n$, it is necessary that all its coefficients are real and sufficient that all are nonnegative. Loewy and London noted that there are polynomials in $\mathcal{P}_n$ with some negative coefficients. It is desirable to characterize $\mathcal{P}_n$. This is known only for $n = 1$ and $n = 2$, where the latter has been recently obtained by Clark and Paparella.

It is straightforward to see that for, any positive integer n, $\mathcal{P}_{n+1} \subseteq \mathcal{P}_n$. Clark and Paparella showed that $\mathcal{P}_2 \subset \mathcal{P}_1$ and $\mathcal{P}_3 \subset \mathcal{P}_2$, and raised the following conjecture:

For every positive integer n, $\mathcal{P}_{n+1} \subset \mathcal{P}_n$.

In this talk we prove this conjecture.

Bisymmetric Nonnegative Jacobi Matrix Realizations

A.M. Encinas

Universitat Politècnica de Catalunya (UPC)

The spectral theory of Jacobi matrices; i.e., real irreducible symmetric matrices $J(a, b)$ with main diagonal $a = (a_1, \ldots, a_n)$ and second diagonal $b = (b_1, \ldots, b_{n-1})$, $b > 0$, is nowadays a well-developed area of linear algebra and functional analysis. The inverse eigenvalue problems for Jacobi matrices have also been studied in great detail in different ways, see for instance [1, 4, 5, 6]. The case concerning to bisymmetric Jacobi matrices has always deserve special attention because its dynamic interpretation as mass-spring chains with symmetrically distributed beads, see [3],[5].

H. Hochstadt proved that given an ordered list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, $\lambda_1 > \cdots > \lambda_m$, there exists at most one bisymmetric Jacobi matrix $J(a, b)$ realicing Λ, see [6, Theorem 3] and in [5, Theorem 3], O.H. Hald announced without proof that such matrix exists (such a proof can be found in [1, Section 3] where the raised problem is named as Problem C). We emphasize that in general such a matrix is not non-negative. For this, it is necessary the list satisfies $\lambda_k + \lambda_{n+1-k} \geq 0$, for any $k = 1, \ldots, n$. Additional hypotheses, as $\lambda_k + \lambda_{n+1-k} > 0$ or $\lambda_k + \lambda_{n+1-k} = 0$ for any $k = 1, \ldots, n$, assures the realization by a nonnegative and irreducible Jacobi matrix. However, neither of the above conditions guarantees that the realizing matrix is bisymmetric in addition.

We focus here in studying the realizability of a given ordered list by a bisymmetric Jacobi matrix. We first apply the structural properties of bisymmetric matrices, see [2] to Jacobi matrices to reduce the raised problem to another one of half size. In this way we can characterize the spectra of nonnegative irreducible bisymmetric Jacobi matrices of low size and also give the unique entries of the matrix in terms of the eigenvalues.

This is joint work with M.J. Jiménez (UPC), C. Marijuán (Universidad de Va lladolid, UVa), M. Mitjana (UPC) and M. Pisonero (UVa). Partially supported by the Departament de Matemàtiques (UPC).

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A combinatorial characterization of lists realizable by compensation in the SNIEP

Julio Moro

Universidad Carlos III de Madrid

The SNIEP (Symmetric Nonnegative Inverse Eigenvalue Problem) deals with characterizing the possible spectra of symmetric entrywise nonnegative matrices. Any list of real numbers which is the spectrum of such a matrix is said to be realizable. Among all realizable lists a subclass has been identified as those 'realizable by compensation' (in short, C-realizable), which was shown in [1] to include most of subclasses known so far associated with sufficient realizability conditions.

In this talk we present a combinatorial characterization of C-realizable lists, first for the special case of zero-sum lists [2], and then for arbitrary ones with nonnegative sum. One of the consequences of this characterization is that the set of zero-sum C-realizable lists is the union of polyhedral cones whose faces are described by equations involving only linear combinations with coefficients 1 and -1 of the entries in the list. Lists with positive sum are C-realizable if and only if there exists a shifted version with zero sum satisfying the equations mentioned above.

This is joint work with Carlos Marijuán (Universidad de Valladolid (Spain)). Supported by the Spanish Ministerio de Economía y Competitividad under grants PGC2018-096446-B-C21, MTM2017-84098-P and MTM2017-90682-REDT.

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MS-3: Copositive and completely positive matrices and related topics

Organisers: Avi Berman (Technion, Haifa), Mirjam Dür (University of Augsburg) and Naomi Shaked-Monderer (Max Stern Yezreel Valley College)

Theme: The concept of copositivity can be traced back to Theodore Motzkin in 1952, and that of complete positivity to Marshal Hall Jr. in 1958. The two classes of matrices are related, and both have received considerable attention in the linear algebra community over the years, and in the last two decades also in the mathematical optimization community. They also arise naturally in various applications. In this minisymposium we bring together people working on these classes of matrices from all these angles: linear algebra, optimization, and other applications, such as machine learning and quantum information.

Completely positive factorizations associated with Euclidean distance matrices corresponding to an arithmetic progression

Damjana Kokol Bukovšek

University of Ljubljana

Euclidean distance matrices corresponding to an arithmetic progression have rich spectral and structural properties. We exploit those properties to develop completely positive factorizations of translations of those matrices. We show that the minimal translation that makes such a matrix positive semidefinite results in a completely positive matrix. We also discuss completely positive factorizations of such matrices over the integers. Methods developed can be used to find completely positive factorizations of other matrices with similar properties.

This is joint work with Thomas Laffey (University College Dublin) and Helena Šmigoc (University College Dublin).

Symmetric Nonnegative Trifactorization Rank

Helena Šmigoc

University College Dublin

The Symmetric Nonnegative Matrix Trifactorization (SNT-factorization) is a factorization of an $n \times n$ nonnegative symmetric matrix A of the form BCB^T, where C is a $k \times k$ symmetric matrix, and both B and C are required to be nonnegative. SNT-factorization is a special case of nonnegative matrix factorization, as well as a generalization of the completely positive factorization. In this talk we define and present some basic properties of the the SNT-rank of A, defined as the minimal k, for which a factorization described above exists. We will compare the ST-rank with the completely positive rank.

This is joint work with Damjana Kokol Bukovšek (University of Ljubljana).

The Maximal Angle between 5×5 Positive Semidefinite and 5×5 Non-negative matrices

Qinghong Zhang

Northern Michigan University

Hiriart-Urruty and Seeger in 2010 conjectured that the maximal angle for two $n \times n$ copositive matrices is $\frac{3\pi}{4}$ for $n \geq 3$. Goldberg and Shaked-Monderer in 2014 disproved the conjecture by constructing a sequence of pairs of matrices. Each pair consists of a positive semidefinite matrix and a non-negative matrix of the same order. The problem of calculating or estimating the maximal angle between an $n \times n$ positive semidefinite matrix and an $n \times n$ non-negative matrix is interesting in its own right as pointed out by Goldberg and Shaked-Monderer. While this problem is completely solved for $n \leq 4$ by Goldberg and Shaked-Monderer, in this study we formulate a signomial geometric programming problem to find the maximal angle between 5×5 semidefinite and 5×5 non-negative matrices. Instead of using an optimization problem solver to solve the problem numerically, we use the method of Lagrange Multipliers to solve the signomial geometric program, and therefore, to find the maximal angle between the cone of 5×5 semidefinite matrices and the cone of 5×5 non-negative matrices.

Factorization of Completely Positive Matrices

Mirjam Dür

Augsburg University

A matrix A is called completely positive, if there exists an entrywise nonnegative matrix B such that $A = BB^T$. These matrices play a major role in combinatorial and quadratic optimization. In this talk we study the problem of finding a nonnegative factorization BB^T of a given completly positive matrix A. We formulate this factorization problem as a nonconvex feasibility problem and develop a solution method based on alternating projections. A local convergence result can be shown for this algorithm. We also provide a heuristic extension which improves the numerical performance of the algorithm. Extensive numerical tests show that the factorization method is very fast in most of the test instances.

This is joint work with Patrick Groetzner.

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On the algebraic structure of the copositive cone

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A closed convex cone can be decomposed into a disjoint union of interiors of its faces. This well-known facial decomposition yields a lot of information on the structure of the cone. However, in general there are infinitely many faces, and for some purposes this decomposition is too fine. Some cones admit a coarser, finite decomposition which unites faces which are of the same type. For example, the cone of positive semi-definite matrices of size n decomposes into $n+1$ relatively open manifolds, each of which contains positive semi-definite matrices of constant rank and which are themselves unions of interiors of similar faces.

We propose such a finite decomposition for the copositive cone $\mathcal{COP}^n$. The components of the decomposition are parameterized by the *extended minimal zero support set*. This means that each component $S_\mathcal{E}$ is composed of copositive matrices A with the same extended minimal zero support set $\mathcal{E}$. This set is a collection of pairs $\mathcal{E} = (I_\alpha, J_\alpha)_{\alpha=1,\dots,|\mathcal{E}|}$, where α enumerates the minimal zeros u_α of A, I_α is the support of the minimal zero u_α, and the index set $J_\alpha \supset I_\alpha$ consists of those indices $j \in \{1,\dots,n\}$ such that $(Au_\alpha)_j = 0$.

The set $S_\mathcal{E}$ lies in a real-algebraic variety $Z_\mathcal{E}$ which is given by a finite number of polynomial equalities, namely those equivalent to the rank-deficiency of the sub-matrix $A_{I_\alpha \times J_\alpha}$. Our main result states that for every $A \in \mathcal{COP}^n$ with extended minimal zero support set $\mathcal{E}$, there exists a neighbourhood U of A in the space of real symmetric matrices such that $U \cap Z_\mathcal{E} \subset S_\mathcal{E}$, i.e., $S_\mathcal{E}$ is open in $Z_\mathcal{E}$. Thus the polynomial equalities cited above fully determine the local structure of $S_\mathcal{E}$.

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Generating extreme copositive matrices near matrices obtained from COP-irreducible graphs

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In this work we use a recently proposed method of copositive cone decomposition to generate extreme and irreducible copositive matrices. We obtain local conditions for components containing matrices derived from cop-irreducible graphs [7]. For one graph we describe the component completely. We exhibit examples of singular points in the component for some of the graphs. These results and our software can be used to generate extreme copositive matrices on the boundary of the copositive cone to test cone approximations.

A symmetric matrix A is called copositive if $\forall x \in \mathbb{R}^n_+$ we have $x^T Ax \geq 0$, where $\mathbb{R}^n_+$ is the set of all n-dimensional nonnegative vectors. The cone $\mathcal{COP}^n$ of copositive matrices is heavily used in non-convex optimization [1] and in approximate solutions of combinatorial optimization problems [2].

In [6] R. Hildebrand has proposed a method of decomposing the copositive cone into a disjoint union of relatively open subsets, each containing matrices with similar extended minimal zero support set. P. Dickinson, R. de Zeeuw. in [7] have proposed a method of extreme and irreducible matrix generation based on cop-irreducible graphs. We derive conditions describing the components containing these matrices, thus expanding the scope of available special copositive matrices for testing approximations of the copositive cone.

For cop-irreducible graphs with stability number 3 we get a components' trigonometric parametrization with linear conditions on the angles. For the C_7 graph we derive global conditions which characterize the component completely. For most of the graphs with ≤ 10 vertices we give a local description of the component and for some graphs their component contains a singularity in the central point. The software we provide can be used to get the dimension and a local description of components for any cop-irreducible graph with stability number 3.

This is joint work with Roland Hildebrand (LJK/CNRS), Roman Tarasov and Mikhail Seliugin (Moscow Institute of Physics and Technology)

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Entangled symmetric quantum states and copositive matrices

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Entanglement is one of the most intriguing phenomena in quantum physics whose implications have profound consequences, not only from a theoretical point of view but also in light of some computational tasks that would be otherwise unfeasible with classical systems. For this reason, deciding whether a quantum state is entangled or not, is a problem of paramount importance whose solution, unfortunately, is known to be NP-hard in the general scenario. In some cases, however, symmetries provide a useful framework to recast the separability problem in a simpler way, thus reducing the original complexity of this task.

In this work we focus on symmetric quantum states, i.e., states that are invariant under permutations of the parties, showing how, in the case of the qudits, the characterization of the entanglement can be accomplished by means of copositive matrices [1]. In particular, we establish a connection between entanglement witnesses, i.e., hermitian operators that are able to detect entanglement, and copositive matrices, showing how only a subset of them, dubbed as exceptional, can be used to assess a non-trivial form of entanglement, so-called PPT-entanglement, in any dimension, with the PPT-entangled edge states detected by the so-called extremal matrices.

Finally we illustrate our findings discussing some examples of families of PPT-entangled states in 3-level and 4-level systems, along with the entanglement witnesses that detect them. We conjecture that any PPT-entangled state of two qudits can be detected by means of an entanglement witness of the form that we propose [2].

This is joint work with Albert Aloy (Vienna), Carlo Marconi, Rubén Quesada, Maciej Lewenstein and Anna Sanpera (Barcelona).

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Copositivity and the Riccati Equation

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To date, much of the research related to copositivity in the systems and control literature has focussed on questions of stability and various classes of copositive Lyapunov functions for positive systems. Alongside the Lyapunov equation, the Riccati equation is one of the most widely studied matrix equations in control theory, and plays a key role in the classical linear quadratic regulator (LQR) problem in optimal control. For a general linear time-invariant (LTI) system, the objective function in the LQR problem is defined by positive definite and positive semi-definite matrices. This leads to Riccati equations and inequalities in which the positive semi-definite cone plays a central role. In considering the LQR problem for a positive system, it is natural to consider objective functions given by copositive matrices instead. This leads us to consider Riccati equations and inequalities where the role of the positive semi-definite cone is taken by the copositive cone. In this talk, several classical results will be recalled concerning ordering and extremal solutions of Riccati inequalities and equations with respect to the partial order defined by the positive semi-definite cone. A number of problems concerning Riccati equations with copositive coefficients will be discussed, and comparison theorems, results on extremal solutions, and on the existence of copositive solutions will be described for this case. The relationship with the LQR problem for positive systems will also be discussed.

The $\{+,-,0\}$ sign patterns of inverse doubly nonnegative matrices and inverse completely positive matrices

Naomi Shaked-Monderer

The Max Stern Yezreel Valley College

We identify all possible $\{+,-,0\}$ sign patterns of inverse doubly nonnegative (DNN) matrices, and of all inverse completely positive (CP) matrices. We prove that all inverses of DNN realizations of a connected graph share the same $\{+,-,0\}$ sign pattern if and only if the graph is bipartite, and the same holds in CP case. In the DNN case, the characterization generalizes a result of [1] regarding the $\{+,-\}$ sign pattern of inverse DNN matrices, where $+$ denotes a nonnegative entry, and the second result answers a question left open there. We also consider the reverse question: which $\{+,-,0\}$ sign patterns of inverse DNN/CP matrices determine uniquely the graph of their originating DNN/CP matrix. We answer the question in the DNN case, but the CP case is still open.

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Linear preservers of copositive and completely positive matrices

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A linear preserver is a linear map L on a space of matrices that preserves a subset K or a relation $\mathcal{R}$. There are two types of preserver problems. The first one, called strong/onto preservers, is to determine the structure of a map L defined on a space of matrices such that $L(K) = K$. The other one is to determine the structure of L such that $L(K) \subset K$. These are called *into* preservers. Linear preservers of the closed convex cone of copositive matrices, COP_n, and its dual, CP_n, pose interesting questions. One may refer to [2] for details. Strong linear preservers of these cones are completely characterized in [3, 4]. However, a complete answer to the *into* linear preservers of either of these cones remains unsolved. This presentation concerns deriving the structure of an invertible linear map on $\mathcal{S}^2$ (the space of real symmetric matrices) such that $L(CP_2) \subset CP_2$. The proof uses a characterization of nonnegativity relative to proper cones from [1].

This is a joint work with Dr. Vatsalkumar Mer, Department of Mathematics, Chungbuk National University, Korea.

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MS-4: Mathematics of quantum information

Organisers: Rupert Levene (University College Dublin) and Ivan Todorov (University of Delaware)

Asymptotic survival of genuine multipartite entanglement in noisy quantum networks depends on the topology

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The study of entanglement in multipartite quantum states plays a major role in quantum information theory and genuine multipartite entanglement signals one of its strongest forms for applications; however, its characterization for general (mixed) states is a highly nontrivial problem. Motivated by the formidable experimental challenge of controlling quantum states with many constituents, we introduce a particularly simple subclass of multipartite states, which we term *pair-entangled network* (PEN) states, as those that can be created by distributing exclusively bipartite entanglement in a connected network, and we study how their entanglement properties are affected by noise and the geometry of the graph that provides the connection pattern. We show that genuine multipartite entanglement in a PEN state depends both on the level of noise and the network topology and, in sharp contrast to the case of pure states, it is not guaranteed by the mere distribution of mixed bipartite entangled states. Our main result, however, is a much more drastic feature of this phenomenon: the amount of connectivity in the network determines whether genuine multipartite entanglement is robust to noise for any system size or whether it is completely washed out under the slightest form of noise for a sufficiently large number of parties. This latter case implies fundamental limitations for the application of certain networks in realistic scenarios, where the presence of some form of noise is unavoidable.

This is joint work with Patricia Contreras-Tejada (ICMAT Madrid) and Carlos Palazuelos (Universidad Complutense de Madrid). Financial support by the Spanish Agencia Estatal de Investigación, Ministerio de Ciencia e Innovación (Grant No. PID2020-113523GB-I00) and by the Comunidad de Madrid (Grant No. QUITEMAD-CMS2018/TCS-4342 and EPUC3M23).

Entanglement annihilation between cones

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Every multipartite entangled quantum state becomes fully separable after an entanglement breaking quantum channel acted locally on each of its subsystems. Whether there are other quantum channels with this property is an open problem with important implications for quantum information theory. I will explain how to cast this problem in the general setting of convex cones in finite-dimensional vector spaces. The entanglement annihilating maps transform the k-fold maximal tensor product of a cone C into the k-fold minimal tensor product of a cone C', and the pair (C, C') is called resilient if all entanglement annihilating maps are entanglement breaking. Using a connection to Banach space tensor norms and solutions to the Hurwitz matrix equations, I will show that the pair (C, C') is resilient when either C or C' is a Lorentz cone. Finally, I will mention some open problems.

This is joint work with Guillaume Aubrun (Lyon). This work was supported in part by ANR (France) under the grant ESQuisses (ANR-20-CE47-0014-01) and by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Action TIPTOP (grant no. 843414.

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Mutually unbiased bases: polynomial optimization and symmetry

Sander Gribling

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A set of k orthonormal bases of $\mathbb{C}^d$ is called mutually unbiased if $|\langle e, f\rangle|^2 = 1/d$ whenever e and f are basis vectors in distinct bases. A natural question is for which pairs (d, k) there exist k mutually unbiased bases in dimension d. The (well-known) upper bound $k \leq d+1$ is attained when d is a power of a prime. For all other dimensions it is an open problem whether the bound can be attained. Navascués, Pironio, and Acín showed how to reformulate the existence question in terms of the existence of a certain C^*-algebra. This naturally leads to a noncommutative polynomial optimization problem and an associated hierarchy of semidefinite programs. The problem has a symmetry coming from the wreath product of S_d and S_k.

We exploit this symmetry (analytically) to reduce the size of the semidefinite programs making them (numerically) tractable. A key step is a novel explicit decomposition of the $S_d \wr S_k$-module $\mathbb{C}^{([d]\times[k])^t}$ into irreducible modules. We present numerical results for small d, k and low levels of the hierarchy. In particular, we obtain sum-of-squares proofs for the (well-known) fact that there do not exist $d+2$ mutually unbiased bases in dimensions $d = 2, 3, 4, 5, 6, 7, 8$.

This is joint work with Sven Polak (CWI Amsterdam).

An Extension of Bravyi-Smolin's Construction for UMEBs

Mizanur Rahaman

Ecole Normale Supérieure de Lyon

Motivated by the concept of Unextendible Product Bases (UPBs), S. Bravyi and J. Smolin introduced the concept of Unextendible Maximally Entangled Bases (UMEBs). This is a collection of orthogonal maximally entangled states in a bipartite system $\mathbb{C}^d \otimes \mathbb{C}^d$ such that there is no maximally entangled state in the orthogonal complement of this set. UMEBs exhibit many interesting features related to entanglement, quantum measurements, Mutually Unbiased Bases etc. In their paper where they introduced UMEBs, Bravyi-Smolin put forward a construction to produce UMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^3$ from a set of equiangular lines in $\mathbb{C}^3$. In this work we extend this construction and show that equiangular subspaces also exhibit examples of UMEBs. These type of projections arise in the context of optimal subspace packing in Grassmannian spaces. This generalization yields new examples of UMEBs in infinitely many dimensions of the underlying system. Consequently, we find orthogonal unitary bases for symmetric subspaces of complex matrices in odd dimensions. This finding validates a recent conjecture about the mixed-unitary rank of the symmetric Werner–Holevo channel in infinitely many dimensions.

This is a joint work with Jeremy Levick (IQC, Waterloo and Univ. of Guelph)

Some results and problems in Quantum Tomography

Chi-Kwong Li

College of William & Mary

Recent results and questions in quantum state and quantum process tomography will be presented. Some mathematical problems related to the implementations of the schemes using different computing platforms such as IBMQ, NMR, optics, will be discussed.

Central limit theorems for braided coin tosses

Claus Koestler

UCC - National University of Ireland

We consider certain representations of the infinite braid group on the infinite tensor product of complex 2×2-matrices, to set up braided sequences of quantum coin tosses. We show that such sequences provide central limit laws in quantum probability which interpolate between the normal distribution and the symmetric Bernoulli distribution. We establish explicit moment formulas for these laws through the combinatorics of directed ordered pair partitions.

This is joint work with Ayman Alahmade (Taibah University).

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Evaluating Quantum Instruments

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Whenever physically implementing a quantum measurement, it is always necessary to accurately evaluate it in comparison to its ideal implementation. A useful way of representing measurements is by quantum instruments: completely positive trace preserving maps that send a quantum state (density matrix) to a mixed state consisting of possible measurement outcomes and their post-measurement state. And so the question becomes: what is an appropriate figure of merit to compare quantum instruments? In this talk we will be reviewing the framework of quantum instruments and exploring two figures of merit, the process fidelity and diamond distance, that can be used to evaluate them.

This is joint work with Joel J. Wallman (University of Waterloo, and Keysight Technologies)

Universal operator systems generated by projections

Travis B. Russell

Department of Mathematics, Dartmouth College

We describe explicit constructions for finite-dimensional operator systems generated by projections satisfying certain linear relations. In particular, we describe operator systems spanned by products of commuting projection-valued measures. At the first matrix level, the ordered vector space constructed satisfies the property that its state space is affinely homeomorphic to the set of quantum-commuting correlations. We discuss corresponding constructions for local, quantum, and quantum-approximate correlations, and implications for the recently discovered separation of the correlation sets and the resolution of Connes' embedding problem.

This is joint work with Roy Araiza (University of Illinois Urbana-Champaign) and Mark Tomforde (University of Colorado Colorado Springs).

Quantum Advantage in Information Retrieval

Mark Howard

NUI Galway

Quantum systems offer advantages over classical ones for various types of information processing. Here [1] we show a quantum-over-classical advantage for a task we call information retrieval. We demonstrate this with a battleshiplike game we call the Torpedo Game. Alice and Bob, finding themselves on opposing sides in a naval conflict wish to subvert their orders while not directly disobeying them, with the goal of avoiding casualties. To do this Alice is allowed very limited communication with Bob, who must retrieve enough information from the message about Alice's whereabouts to avoid sinking her ship. With quantum communication perfect strategies are possible, something that is not achievable with classical communication only.

Quantum systems can outperform classical ones in a variety of information-processing tasks. However, the precise features of quantum systems that enable information processing advantages are not fully understood. For information retrieval tasks we pinpoint a feature known as contextuality, which relates to classical logical paradoxes as being at the root of quantum advantage, and moreover show that the degree of contextuality present quantifies the degree of advantage that can be obtained.

Our insight into the source of quantum advantage in information retrieval, and the broad approach we develop in this work for treating information retrieval in quantum settings, have led us to propose the Torpedo Game, but can also lend themselves to discovering further protocols exhibiting quantum advantage. The example of the Torpedo Game also relies on an experimentally accessible three-level system, that make this work amenable to implementation with current technology.

This is joint work with Pierre-Emmanuel Emeriau and Shane Mansfield (Quandela SAS, Paris). M.H. is supported by a Royal Society-Science Foundation Ireland University Research Fellowship.

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Matrices of interest in higher dimensional quantum walks

Michael Mc Gettrick

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In discrete quantum walks, we are interested in quantum "coin" operations defined by choosing a matrix from SU(n). The canonical example in the simplest classical case (one dimensional walk) is defined using a coin with two faces, which can be in "heads" or "tails" state, and with only two possible operations (Identity operation or **NOT** operation). For the corresponding quantum case, we have three continuous parameters to choose to fix our SU(2) matrix. If we want to execute a quantum walk on a high dimensional lattice, or high degree graph, analysis becomes difficult because of physics problems (controlling high dimensional quantum states) and mathematics problems (the parameter choice grows quadratically with n).

In this talk we will describe some specific matrices that arise in high dimensional quantum walks. Amongst our examples are two families:

- *Quantum walks on the integer lattice* $\mathbb{Z}^n$*.* The direct way of defining such walks is to choose an element of SU($2n$). Using an alternating walk, we can create a subset of quantum walks on $\mathbb{Z}^n$ by just choosing an SU(2) matrix (reducing the parameter choice from 15 to 3, for example, in the case of the square lattice).
- *Quantum walks with memory ("history").* These are analogous to higher order Markov chains. To define such a walk with m memory steps on a graph G of degree d means choosing a matrix in SU(d^m). We show this can be re-defined as a quantum walk *without* memory on the line graph $L^m(G)$.

Quantum Information: the Mathematics behind the quantification of quantum entanglement and the distinction of quantum states

Victoria Sánchez Muñoz

National University of Ireland Galway

In the last few decades there has been an increasing interest in Quantum Information, in part due to the experimental improvement of quantum technology. The field of Quantum Information concerns with studying and implementing the transmission of information using quantum mechanical resources, and thus, it makes use of the mathematical framework of Quantum Mechanics and information theory. See [1] for a general outlook of certain important features (and challenges) that Quantum Information possesses due to its quantum mechanical nature.

In the first part of my talk, I will introduce some of the mathematical tools and concepts used in Quantum Information and their physical meaning. Specifically, I will speak about how one of the most important concepts of Quantum Mechanics **-entanglement-** is **quantified**; and also the problem of **distinguishing** two **quantum states**, which is crucial for characterising how well a quantum channel preserves information and for quantum error correction (see [2]). In the second part of my talk, I will give an overview of an ongoing work that uses both concepts -the measurement of entanglement and the distance between quantum states- in the context of **Quantum Games**.

This is an ongoing joint work with Michael Mc Gettrick (NUI Galway). Supported by the College of Science and Engineering at the National University of Ireland Galway.

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Isoperimetric inequalities for quantum graphs

J. Alejandro Chávez-Domínguez

University of Oklahoma

For classical graphs, Cheeger's inequality shows the equivalence between the spectral gap of the Laplacian and the notion of expansion, where the former measures the distance between eigenvalues of a matrix, and the latter quantifies isoperimetric inequalities in the graph.

In the quantum setting, quantum expanders were defined using the spectral gap approach and they have received a significant amount of attention. Previous work of Temme, Kastoryano, Ruskai, Wolf, and Verstraete has already related quantum expanders to an inequality of an isoperimetric flavor, which can be understood as a quantum version of an edge-isoperimetric inequality. In this work, we prove a version of a vertex-isoperimetric inequality for quantum expanders. Our approach is based on the definition of quantum metric spaces of Kuperberg and Weaver. As an application, we prove a quantum version of a classical theorem stating that a metric space that equi-coarsely contains a sequence of expanders must have infinite asymptotic dimension.

This is joint work with Andrew Swift. Supported by NSF grant DMS-1900985.

MS-5: Combinatorial matrix theory

Organisers: Jane Breen (Ontario Tech University) and Roberto Canogar (Universidad Nacional de Educación a Distancia, Madrid)

Two conjectures on the spread of graphs

Michael Tait

Villanova University

Given a graph G let λ_1 and λ_n be the maximum and minimum eigenvalues of its adjacency matrix and define the spread of G to be $\lambda_1 - \lambda_n$. In this talk we discuss solutions to a pair of 20-year-old conjectures of Gregory, Hershkowitz, and Kirkland regarding the spread of graphs.

The first, referred to as the spread conjecture, states that over all graphs on n vertices the join of a clique of order $\lfloor 2n/3 \rfloor$ and an independent set of order $\lceil n/3 \rceil$ is the unique graph with maximum spread. The second, referred to as the bipartite spread conjecture, says that for any fixed $e \leq n^2/4$, if G has maximum spread over all n-vertex graphs with e edges, then G must be bipartite.

We show that the spread conjecture is true for all sufficiently large n, and we prove an asymptotic version of the bipartite spread conjecture. Furthermore, we exhibit an infinite family of counterexamples to the bipartite spread conjecture which shows that our asymptotic solution is tight up to a multiplicative factor in the error term.

This is joint work with Jane Breen, Alex Riasanovsky, and John Urschel.

Algebraic Connectivity and the Laplacian Spread

Mark Kempton

Brigham Young University

The Laplacian Spread Conjecture states that if a graph on n vertices has Laplacian eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ then

$$\lambda_n - \lambda_2 \leq n - 1.$$

By a well-known relationship between the Laplacian eigenvalues of a graph and its complement, the Spread Conjecture can be seen to be equivalent to the statement

$$\lambda_2(G) + \lambda_2(G^c) \geq 1.$$

The second smallest eigenvalue of the Laplacian of a graph is known as its *algebraic connectivity*, and is known to be closely related to how well-connected the graph is. Thus the Spread Conjecture can be interpreted as quantifying how poorly connected both a graph and its complement can possibly be.

We will present a new conjecture on a lower bound for the algebraic connectivity using the eccentricity of vertices in the graph. This will give a new approach to studying the Laplacian spread of a graph and lead to a strengthening of the Laplacian Spread Conjecture.

This is joint work with Wayne Barrett (BYU), Emily Evans (BYU), and Tracy Hall (Hall Labs, LLC).

Spectral Moore Theorems for Graphs and Hypergraphs

Sebastian M. Cioabă

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The spectrum of a graph is closely related to many graph parameters. In particular, the spectral gap of a regular graph which is the difference between its valency and second eigenvalue, is widely seen an algebraic measure of connectivity and plays a key role in the theory of expander and Ramanujan graphs. In this paper, I will give an overview of recent work studying the maximum order of a regular graph (bipartite graph or hypergraph) of given valency whose second largest eigenvalue is at most a given value. This problem can be seen as a spectral Moore problem and has close connections to Alon-Boppana theorems for graphs and hypergraphs and with the usual Moore or degree-diameter problem.

Keywords: Eigenvalues, Alon-Boppana theorem, Ramanujan graphs, spectral Moore bound

This is joint work with Jack Koolen, Masato Mimura, Hiroshi Nozaki, Takayuki Okuda and Jason Vermette.

Oriented Cayley graphs with all eigenvalues being rational multiples of each other

Xiaohong Zhang

University of Waterloo

Let G be a finite abelian group. A Cayley graph on G is a Cayley digraph $X(G, C)$ such that $C = -C$. Bridges and Mena gave a characterization of when a Cayley graph has only integer eigenvalues in 1982 [1]. Here we consider oriented Cayley graph on G, a Cayley digraph $X(G, C)$ such that $C \cap (-C) = \emptyset$, and its $(0, 1, -1)$ skew-symmetric adjacency matrix. We give a characterization of when all the eigenvalues of X are integer multiples of $\sqrt{\Delta}$ for some square-free integer $\Delta < 0$. This also characterizes oriented Cayley graphs on which the continuous quantum walks are periodic, a necessary condition for the walk to admit uniform mixing or perfect state transfer.

This is joint work with Chris Godsil (Waterloo)

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Alternating sign matrices of finite multiplicative order

Rachel Quinlan

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An alternating sign matrix (ASM) is a square $(0, 1, -1)$-matrix in which the non-zero entries alternate in each row and column, beginning and ending with 1. Examples of ASMs include permutation matrices, and there are contexts in which the set of $n \times n$ ASMs may be seen as a natural extension or completion of the set of permutation matrices. Unlike the permutation matrices which form a group, the ASMs are not equipped with any apparent algebraic structure, and the permutation matrices are the only ones to generate cyclic groups whose elements are all ASMs. Nevertheless, there exist (non-permutation) $n \times n$ ASMs that have finite multiplicative order, and that have have finite orders not occurring in the symmetric group of degree n.

We investigate alternating sign matrices that are not permutation matrices, but have finite order in a general linear group. We classify all such examples of the form $P + T$, where P is a permutation matrix and T has four non-zero entries, forming a square with entries 1 and -1 in each row and column. We show that the multiplicative orders of these matrices do not always coincide with those of permutation matrices of the same size. We pose the problem of identifying finite subgroups of general linear groups that are generated by alternating sign matrices.

This is joint work with Cian O'Brien (Cardiff University)

Comparability and cocomparability bigraphs

Jephian C.-H. Lin

National Sun Yat-sen University

Let $\mathcal{F}$ be a family of $0,1$-matrices. A $0,1$-matrix M is symmetrically $\mathcal{F}$-free if there is a permutation matrix P such that $P^\top MP$ does not contain any $S \in \mathcal{F}$ as a submatrix. For a given graph G, the neighborhood matrix of G is defined as $A(G) + I$, where $A(G)$ is the adjacency matrix and I is the identity matrix. Several important graph classes are known to have a characterization from the matrix point of view. For example, let

$$\Gamma = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \texttt{slash} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1}$$

Thus, the strongly chordal graphs are the graphs whose neighborhood matrix is symmetrically $\{\Gamma\}$-free; the cocomparability graphs are the graphs whose neighborhood matrix can be permuted to avoid `slash` on the main diagonal; and the interval graphs are the graphs whose neighborhood matrix is symmetrically $\{\Gamma, \texttt{slash}\}$-free. Note that the set of interval graphs is the intersection of the set of strongly chordal graphs and the set of cocomparability graphs. There are bipartite analogues for the strongly chordal graphs and the interval graphs, namely, the bipartite chordal graphs and the interval containment bigraphs. In this talk, we introduce the cocomparability bigraphs from the matrix perspective as a bipartite analogue to the cocomparability graphs.

This is joint work with Pavol Hell (Simon Fraser University), Jing Huang (University of Victoria), and Ross M. McConnell (Colorado State University).

Spectral restrictions for certain symmetric ±1-matrices with applications to equiangular lines

Gary Greaves

Nanyang Technological University, Singapore

Given some dimension d, what is the maximum number, $N(d)$, of lines in $\mathbb{R}^d$ such that the angle between any pair of lines is constant? (Such a system of lines is called "equiangular".) This classical problem was initiated by Haantjes in 1948 in the context of elliptic geometry. In 1966, Van Lint and Seidel showed certain symmetric $\{\pm 1\}$-matrices, called Seidel matrices, can be associated to an equiangular line system. Up until 2021, $N(14)$ was the smallest unknown value of the sequence $(N(d))_{d\in\mathbb{N}}$.

In this talk, I will present a recently discovered restriction on the characteristic polynomial of Seidel matrices has enabled us to determine the sequence $(N(d))_{d\in\mathbb{N}}$ all the way up to $d = 17$.

This talk is based on joint work with Jeven Syatriadi (Nanyang Technological University) and Pavlo Yatsyna (Charles University).

Orthogonality for $(0, -1)$ tropical normal matrices

M.J. de la Puente

Universidad Complutense

We study pairs (A, B) of order n real matrices operated with tropical sum $\oplus = \max$ and multiplication $\odot = +$ (multiplication symbol omitted in the sequel). A pair is *orthogonal* if $AB = Z_n = BA$, where Z_n is the all zero matrix. Restriction to the *semiring* (ordered, additively idempotent) R of *normal matrices* (i.e., real non–negative matrices with null diagonal) makes the problem more meaningful. Restriction to Boolean normal matrices (i.e., matrices over the (ordered, additively idempotent) semiring $R = \{0, -1\}$) makes the problem combinatorial.

The more zeros in a pair (A, B), the more likely that $AB = Z_n = BA$ happens. We prove that the minimal number of zeros in an orthogonal pair is $4n - 6$, for $n \geq 7$. Pairs attaining this minimum happen in four types. This orthogonality binary relation is also studied in terms of relation graphs.

This is joint work with Bakhad Bakhadly (Moscow) and Alexander Guterman (Moscow). Supported by PID2019-107701GB-I00, Ministerio de Ciencia e Innovación and 910444 Grupo UCM.

Combinatorial Perron Parameters and Classes of Trees

Enide Andrade

Department of Mathematics, University of Aveiro, Portugal

The main goal of this talk is to present recent results related with the combinatorial Perron parameters introduced in [1,2] for certain classes of trees, and related bounds for these parameters. These parameters are related to algebraic connectivity of trees and corresponding centers.

This is joint work with Lorenzo Ciardo (University of Oxford) and Geir Dahl (University of Oslo). Enide Andrade is supported by Center for Research and Development in Mathematics and Applications (CIDMA) through the Portuguese Foundation for Science and Technology (FCT - Fundação para a Ciência e a Tecnologia), UIDB/04106/2020 and UIDP/04106/2020.

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Kemeny's constant for a chain of connected graphs with respect to a tree

Sooyeong Kim

Università di Pisa

In this talk, I provide a formula, with a sketch of the proof, for Kemeny's constant for a graph with bridges, in terms of quantities inherent to the subgraphs upon removal of all bridges: resistance matrices, degree vectors, and the numbers of edges. With the formula, I present several optimization problems for Kemeny's constant for graphs with bridges, and answer some of the problems. Finally, I remark some potential applications regarding the optimization problems and computation.

This is joint work with Jane Breen (Ontario Tech University) and Emanuele Crisostomi (Università di Pisa). Supported by the Research Project PRIN 2017 "Advanced Network Control of Future Smart Grids" funded by the Italian Ministry of University and Research (2020–2023).

Minimum number of distinct eigenvalues allowed by a sign pattern

Minerva Catral

Xavier University

For a real square matrix A, $q(A)$ denotes the number of distinct eigenvalues of A. Sign pattern $\mathcal{A}$ is a square matrix with entries in $\{+,-,0\}$. We introduce the study of the minimum possible value of $q(A)$ over all matrices A with sign pattern $\mathcal{A}$. This minimum value is denoted $q(\mathcal{A})$. We explore $q(\mathcal{A})$ using digraph properties of the sign pattern, and characterize $q(\mathcal{A})$ for small order sign patterns.

This is joint work with J. Breen, C. Brouwer, M. Cavers, P. van den Driessche and K. Vander Meulen.

MS-6: The inverse eigenvalue problem for graphs

Organisers: Jephian Lin (National Sun Yat-sen University, Taiwan) and Polona Oblak (University of Ljubljiana)

Theme: A generalized adjacency matrix of a graph is a symmetric matrix whose off-diagonal entry is nonzero if and only if it corresponds to an edge of the graph, while the diagonal entries can be chosen as any real number. The inverse eigenvalue problem for graphs (IEPG) studies the generalized adjacency matrices of a given graph and aims to find the possible spectra of them. Various related questions can be asked: What is the maximum nullity over all generalized adjacency matrices, and what is the minimum rank? What is the minimum number of distinct eigenvalues? Recently, new techniques, called the strong properties, are developed using the implicit function theorem and have found significant applications to the IEPG.
The minisymposium will present recent progress and open problems in IEPG.

On the maximum multiplicity of the kth largest eigenvalue of a graph.

Shaun Fallat

University of Regina

Given a graph G, we are interested in studying the maximum nullity over all real symmetric matrices $S(G)$ constrained by a fixed number of negative eigenvalues. For the case of trees we re-derive a formula for this maximum nullity and completely describe its behaviour as a function of the number of negative eigenvalues. We build on this analysis by presenting an analogous result for unicyclic graphs and verifying a surprising relation between this maximum nullity and a 2-player version of zero forcing for threshold graphs.

Part of this work joint with Mohammad Adm (Palestine Polytechnic) and part is joint project with the 2021 DMRG at the University of Regina. Research supported in part by an NSERC Discovery Research Grant, Grant No. RGPIN-2019-03934.

A zero forcing menagerie: the ordered multiplicity inverse eigenvalue sequence problem, powers of graphs, and more

Franklin Kenter

United States Naval Academy

Given a graph G, one may ask: "What sets of eigenvalues are possible over all weighted adjacency matrices of G?" (Here, negative and diagonal weights are allowed). This is known as the Inverse Eigenvalue Problem for Graphs (IEPG) A mild relaxation of this question considers the multiplicity sequence instead of the exact eigenvalues themselves. For instance, given a graph G on n vertices and an ordered partition $\mathbf{m} = (m_1, \ldots, m_\ell)$ of n, is there a weighted adjacency matrix where the i-th distinct eigenvalue has multiplicity m_i? This is known as the ordered multiplicity inverse eigenvalue sequence problem. Recent work has solved this problem for all graphs on 6 vertices.

In this talk, we develop zero forcing methods for the ordered multiplicity IEPG in a multitude of different contexts. Namely, we apply a menagerie of zero forcing parameters on powers of graphs to achieve bounds on sums of various multiplicities. Not only can we verify the above result in a more straight-forward manner, but we apply our techniques to skew-symmetric matrices, nonnegative matrices, among others.

This is joint work with Jephian C.-H. Lin (National Sun Yat-sen University). Supported in part by the National Science Foundation, Grant DMS-1720225, and the Office of Naval Research, Grant ONR-749N0016120WX00637, and Taiwan Ministry of Science and Technology, Grant MOST-109-2536-M-110-006.

The Strong Nullity Interlacing Property

Mary Flagg

University of St. Thomas

Given a graph G, let $\mathcal{S}(G)$ be the set of all real symmetric matrices with graph G. Strong properties have been very useful to assert that if there is a matrix in $\mathcal{S}(G)$, with particular eigenvalue properties, then there exists a matrix in $\mathcal{S}(H)$ with the same properties for any supergraph H on the same vertex set as G.

The Cauchy interlacing inequalities give the relationship between the eigenvalues of a matrix $A \in \mathcal{S}(G)$ and the eigenvalues of its principal submatrix $A(n)$ formed by deleting row and column n, which may be viewed as the matrix for the graph $G - n$ obtained by deleting vertex n. The strong nullity interlacing property is a tool for creating supergraph H of G with the property that there exists a matrix $B \in \mathcal{H}$ such that the nullities of B and $B(n)$ are the same as those of A and $A(n)$, respectively.

This is joint work with Aida Abiad (Maastricht University), Bryan A. Curtis (Iowa State University), H. Tracy Hall (T. Hall LLC), Jephian C.-H. Lin (National Sun Yat-Sen University), Bryan Shader (University of Wyoming), John Sinkovic (Brigham Yound University). This work is partially supported by the American Institute of Mathematics through the IEPG Research Community.

Strong Spectral Norm Property

Bryan Curtis

Iowa State University

A sign pattern is a matrix with entries coming from the set $\{0, 1, -1\}$. The sign pattern of a real matrix is the sign pattern obtained by replacing each positive and negative entry with a 1 and -1, respectively. The class of all real matrices with sign pattern S is denoted $\mathcal{Q}(S)$. For a given sign pattern S, we are interested in what can be said about the singular values of matrices in $\mathcal{Q}(S)$. More specifically, we shall investigate the set of $m \times n$ matrices that have a largest singular value of fixed multiplicity k, denoted $\mathcal{O}(m, n, k)$, and their sign patterns. In this talk we introduce the strong spectral norm property (SSNP) and demonstrate how the SSNP is used to study the sign patterns of matrices in $\mathcal{O}(m, n, k)$.

This is joint work with Bryan Shader (University of Wyoming)

The Allows Problem for Graphs with Two Distinct Eigenvalues

Shahla Nasserasr

Rochester Institute of Technology

For a graph G, the minimum number of distinct eigenvalues over all matrices whose nonzero off-diagonal entries correspond to the edges of G is denoted by $q(G)$. Considering connected graphs G, the allows problem asks how many edges are necessary to allow $q(G) = 2$. In this talk we discuss the current advances on the allows problem.

This is joint work with the AIM Qq Group.

On the number of distinct eigenvalues of joins of two graphs

Polona Oblak

University of Ljubljana

We introduce a combinatorial necessary condition for the join $G \vee H$ of graphs G and H to be the pattern of an orthogonal symmetric matrix, or equivalently, that the minimum number of distinct eigenvalues $q(G \vee H)$ is equal to two. This combinatorial property depends on a notion of compatibility between the possible multiplicity lists for the graphs G and H. In some cases this necessary condition is also sufficient and hence completely resolves the question of when $q(G \vee H) = 2$. We present some special cases and consequences.

This is joint work with Rupert H. Levene and Helena Šmigoc (University College Dublin).

Inverse eigenvalue and related problems for hollow matrices described by graphs

Derek Young

Mount Holyoke College

A hollow matrix described by a graph G is a real symmetric matrix having all diagonal entries equal to zero and with the off-diagonal entries governed by the adjacencies in G. For a given graph G, the determination of all possible spectra of matrices associated with G is the hollow inverse eigenvalue problem for G. In this talk, solutions to the hollow inverse eigenvalue problems for paths and complete bipartite graphs are presented. Results for related subproblems such as possible ordered multiplicity lists, maximum multiplicity of an eigenvalue, and minimum number of distinct eigenvalues are presented for additional families of graphs.

Spectral arbitrariness for trees fails spectacularly, I

Rupert Levene

University College Dublin

If G is a graph and $\mathbf{m}$ is an ordered multiplicity list which is realisable by at least one symmetric matrix with graph G, what can we say about the eigenvalues of all such realising matrices for $\mathbf{m}$? While it is tempting to believe that every set of distinct real eigenvalues should always be realisable (spectral arbitrariness), in [1], F. Barioli and S. Fallat produced the first counterexample: a tree G on 10 vertices and an ordered multiplicity list $\mathbf{m}$ for which every realising set of eigenvalues obeys a nontrivial linear constraint. We extend this by giving an infinite family of trees and ordered multiplicity lists whose sets of realising eigenvalues are very highly constrained, with at most 5 degrees of freedom, regardless of the size of the tree in this family.

This is joint work with, Shaun M. Fallat, H. Tracy Hall, Seth A. Meyer, Shahla Nasserasr, Polona Oblak and Helena Šmigoc.

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Spectral arbitrariness for trees fails spectacularly, II

H. Tracy Hall

Hall Labs, LLC (Provo, UT, USA)

The Inverse Eigenvalue Problem for a Graph asks what spectra are possible for a real symmetric matrix whose pattern of off-diagonal nonzero entries is exactly specified by a given graph G. An important relaxation of this problem asks only which ordered multiplicity lists of eigenvalues are possible. It was thought for a time that, at least in the case where G has no cycles, the two questions might be equivalent—that an achievable ordered list of multiplicities would always be *spectrally arbitrary,* achievable with any prescribed set of gaps bridging from one multiplicity to the next. This early hope was dashed by F. Barioli and S. Fallat, who produced a small counterexample tree whose eigenvalue gaps, for a particular ordered multiplicity list, must satisfy a linear constraint [1].

We show that for a very broad family of trees there exist multiplicity lists whose eigenvalue gaps must satisfy many more, typically non-linear, constraints. The failure of spectral arbitrariness culminates in an example, for any tree in the family of sufficient depth, of a multiplicity list whose relative spacing of eigenvalues is completely rigid.

This is ongoing joint work with Shawn M. Fallat, Rupert Levene, Seth A. Meyer, Shahla Nasserasr, Polona Oblak, and Helena Šmigoc.

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MS-7: General preservers

Organiser: Lajos Molnár (University of Szeged)

Distance-preserving bijections between sets of invertible elements in unital Jordan-Banach algebras

Antonio M. Peralta

Instituto de Matemáticas de la Universidad de Granada (IMAG). Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain.

It is known that the structure of the group, A^{-1}, of invertible elements of a unital Banach algebra A does not determine uniquely the structure of the algebra, there are examples of non-isomorphic unital Banach algebras A and B whose groups of invertible elements are topologically isomorphic. If we also assume a preservation of the metric structure induce by the norm on the group of invertible elements, the answer is different. O. Hatori proved in [1] that, for each surjective isometry Δ from an open subgroup of the group of invertible elements in an associative unital semisimple commutative Banach algebra A onto an open subgroup of the group of invertible elements in an associative unital Banach algebra B, the mapping $\Delta(\mathbf{1})^{-1}\Delta$ is an isometric group isomorphism, which extends to an isometric real-linear algebra isomorphism from A onto B.

In this talk we shall try to understand the general form of a bijection preserving distances between the sets, M^{-1} and N^{-1}, of invertible elements of two unital Jordan-Banach algebras M and N, respectively. In this case, if $\mathfrak{M} \subseteq M^{-1}$ and $\mathfrak{N} \subseteq N^{-1}$ are clopen subsets of M^{-1} and N^{-1}, respectively, which are closed for powers, inverses and products of the form $U_a(b)$, and $\Delta : \mathfrak{M} \to \mathfrak{N}$ is a surjective isometry, then there exists a surjective real-linear isometry $T_0 : M \to N$ and an element u_0 in the McCrimmon radical of N such that $\Delta(a) = T_0(a) + u_0$ for all $a \in \mathfrak{M}$. The conclusion is even more satisfactory in the case of unital JB*-algebras.

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On isometric rigidity of Wasserstein spaces

Tamás Titkos

Alfréd Rényi Institute of Mathematics

Given a metric space (X, r) and a subset $\mathcal{S} \subseteq \mathcal{P}(X)$ of all probability measures, one can endow $\mathcal{S}$ with various metrics, depending on what kind of measurement is suitable for the problem under consideration. For example, the *Kolmogorov-Smirnov metric* d_{KS} on $\mathcal{S} = \mathcal{P}(\mathbb{R})$ is frequently used in statistics to compare a sample with a reference probability distribution. The *Lévy-Prokhorov metric* d_{LP} plays an important theoretical role in several limit theorems in probability theory. In this case, (X, ϱ) is a complete separable metric space, and $\mathcal{S} = \mathcal{P}(X)$. The *quadratic Wasserstein metric* $d_{\mathcal{W}_2}$ turned out to be very effective in a wide range of AI applications including pattern recognition and image processing problems. In these applications, (X, r) is typically the n-dimensional Euclidean space, and $\mathcal{S}$ is the collection of all Borel probability measures with finite second moment.

In recent years, there has been a considerable interest in the characterization of surjective distance preserving maps of the above-mentioned (and many other) metric spaces of measures, see e.g. [1, 2, 3, 4, 4, 5, 6, 7, 8]. In most cases, it turned out that isometries of $\mathcal{S}$ are strongly related to self-maps of the base space.

In this talk, we will describe the structure of isometries in the cases when (X, r) is a separable real Hilbert space [4, 5] or a graph metric space [6], and $\mathcal{S}$ is the collection of all Borel probability measures with finite p-th moment for some $p \geq 1$.

This is joint work with György Pál Gehér, Gergely Kiss, and Dániel Virosztek. Supported by the Momentum Program of the Hungarian Academy of Sciences (grant no. LP2021-15/2021) and by the Hungarian National Research, Development and Innovation Office - NKFIH (grant no. K115383).

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On property ($\mathbb{B}$) and zero product determined Banach algebras

Jerónimo Alaminos

University of Granada

We will briefly survey examples and counterexamples about property ($\mathbb{B}$) and zero product determined Banach algebras.

Automorphisms of effect algebras

Peter Šemrl

University of Ljubljana

There are several relations and operations on effect algebras that are important in mathematical foundations of quantum mechanics. Among them are the usual partial order, coexistency, and orthocomplementation. Automorphisms of effect algebras with respect to these relations and/or operations will be discussed.

Adjacency preserving transformations of conjugacy classes of finite-rank self-adjoint operators

Mark Pankov

University of Warmia and Mazury

Classical Chow's theorem states that bijective transformations of Grassmannians preserving the adjacency relation in both directions are induced by semilinear automorphisms of the corresponding vector spaces and semilinear isomorphisms to the dual vector spaces. Every Hilbert Grassmannian can be naturally identified with a conjugacy class of finite-rank projections. Chow's theorem reformulated in these terms was successfully exploited to prove Wigner-type theorems. We extend the concept of adjacency on conjugacy classes of finite-rank self-adjoint operators (such an extension is not immediate). If operators from such a class have at least three eigenvalues, then every bijective transformation of this class preserving the adjacency relation in both directions is induced by a unitary or anti-unitary operator up to a permutation of eigenspaces with the same dimensions. For conjugacy classes with two eigenvalues the above statement fails.

This is joint work with Krzysztof Petelczyc (Białystok) and Mariusz Żynel (Białystok).

Collineations of a linear transformation

Janko Bračič

University of Ljubljana, Slovenia

Given a linear transformation A on a finite-dimensional complex vector space V, we study the group $\mathrm{Col}(A)$ consisting of those invertible linear transformations S on V for which the mapping Φ_S defined as $\Phi_S\colon \mathcal{M} \mapsto S\mathcal{M}$ is an automorphism of the lattice $\mathrm{Lat}(A)$ of all invariant subspaces of A. By using the primary decomposition of A, we first reduce the problem of characterizing $\mathrm{Col}(A)$ to the problem of characterizing the group $\mathrm{Col}(N)$ of a given nilpotent linear transformation N. While $\mathrm{Col}(N)$ always contains all invertible linear transformations of the commutant $(N)'$ of N, it is always contained in the reflexive cover $\mathrm{AlgLat}(N)'$ of $(N)'$. We prove that $\mathrm{Col}(N)$ is a proper subgroup of $(\mathrm{AlgLat}(N)')^{-1}$ if and only if at least two Jordan blocks in the Jordan decomposition of N are of dimension 2 or more.

This is joint work with Marko Kandić (University of Ljubljana). Supported by the Slovenian Research Agency through the research program P2-0268.

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Preservers of moment sequences

Apoorva Khare

Indian Institute of Science; and Analysis and Probability Research Group (Bangalore, India)

Call a measure on $\mathbb{R}$ *admissible* if it is non-negative and admits all moments. We classify all functions on the real line which when applied termwise, preserve the class of moment-sequences of admissible measures (i.e., take one such sequence to another). We show that all such functions are absolutely monotonic – and conversely – and that surprisingly, it suffices to restrict the test measures to three point masses in $[-1, 1]$. This strengthens and parallels a dimension-free positivity preserver result by Schoenberg [*Duke* 1942] and Rudin [*Duke* 1959], and is joint work with Belton, Guillot, and Putinar [*JEMS*, to appear].

Barycenters of Hellinger distances and Kubo-Ando means as barycenters

Dániel Virosztek

Alféd Rényi Institute of Mathematics, Hungary

The first part of the talk is devoted to quantum Hellinger distances — introduced recently by Bhatia et al. [1] — with a particular emphasis on barycenters. We introduce the family of generalized quantum Hellinger divergences that are of the form $\phi(A,B) = \operatorname{Tr}\left((1-c)A + cB - A\sigma B\right)$, where σ is an arbitrary Kubo-Ando mean, and $c \in (0,1)$ is the weight of σ. We note that these divergences belong to the family of maximal quantum f-divergences, and hence are jointly convex, and satisfy the data processing inequality (DPI). We will present a fixed-point equation that characterizes of the barycenter of finitely many positive definite operators for these generalized quantum Hellinger divergences [3].

In the second part, we present a divergence center interpretation of general symmetric Kubo-Ando means [4]. This characterization of the symmetric means naturally leads to a definition of weighted and multivariate versions of a large class of symmetric Kubo-Ando means. We study elementary properties of these weighted multivariate means, and note in particular that in the special case of the geometric mean we recover the weighted $\mathcal{A}\#\mathcal{H}$-mean introduced by Kim, Lawson, and Lim [2].

This is joint work with József Pitrik (TU Budapest). Virosztek is supported by the Momentum program of the Hungarian Academy of Sciences under grant agreement no. LP2021-15/2021.

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Preservers related to the geometric mean and its variants

Lajos Molnár

University of Szeged and Budapest University of Technology and Economics

We consider positive definite cones in operator algebras equipped with the operation of the usual Kubo-Ando geometric mean or one of its variants (Rényi power mean, Fiedler-Pták spectral geometric mean, log-euclidean mean). We study the precise structures of the corresponding isomorphisms, especially those relating to the Fiedler-Pták spectral geometric mean. The problem concerning their isomorphisms is the one that currently seems to be the most exciting and challenging.

A part of this talk is based on a joint work with Lei Li (Nankai University) and Liguang Wang (Qufu Normal University). The speaker is supported by the Ministry of Innovation and Technology of Hungary from the National Research, Development and Innovation Fund, project no. TKP2021-NVA-09, and also by the National Research, Development and Innovation Office of Hungary, NKFIH, grant no. K134944.

MS-8: Distance matrices of graphs

Organisers: Projesh Nath Choudhury and Apoorva Khare

Theme: Distance matrices associated to graphs have been explored intensively in the literature for several decades now, both from an algebraic and a spectral viewpoint. They have connections to graph embeddings, communications networks, and quantum chemistry among other areas. This minisymposium will bring together researchers working on distance matrices from a variety of perspectives, and discuss modern approaches and recent results.

Extending a conjecture of Graham and Lovász on the distance characteristic polynomial

Aida Abiad

Eindhoven University of Technology, Ghent University, Vrije Universiteit Brussel

Graham and Lovász conjectured in 1978 that the sequence of normalized coefficients of the distance characteristic polynomial of a tree is unimodal with the maximum value occurring at $\lfloor \frac{n}{2} \rfloor$ for a tree T of order n. We extend this old conjecture to block graphs. In particular, we prove the unimodality part and we establish the peak for several extremal cases of block graphs.
This is joint work with B. Brimkov, S. Hayat, A. Khramova and J. Koolen.

Blowup-polynomials of graphs

Projesh Nath Choudhury

Department of Mathematics
Indian Institute of Science, Bangalore

Given a finite simple connected graph $G = (V, E)$ (or even a finite metric space), we introduce a novel invariant which we call its blowup-polynomial $p_G(n_v : v \in V)$. To do so, we compute the determinant of the distance matrix of the graph blowup, obtained by taking n_v copies of the vertex v, and remove an exponential factor. First: we show that as a function of the sizes n_v, p_G is a polynomial, is multi-affine, and is real-stable. Second: we show that the multivariate polynomial p_G fully recovers G. Third: we obtain a novel characterization of the complete multi-partite graphs, as precisely those whose "homogenized" blowup-polynomials are Lorentzian/strongly Rayleigh.

Joint with Apoorva Khare.

Distance ideals of graphs

Carlos A. Alfaro

Banco de México

Distance ideals of graphs generalize, among other graph paramethers, the spectrum and the Smith normal form (SNF) of distance and distance Laplacian matrices. In particular, they allow us to introduce the notion of codeterminantal graphs, which generalize the concepts of cospectral and coinvariant graphs. We show computational results on codeterminantal graphs up to 9 vertices. Although the spectrum of several graph matrices has been widely used to determine graphs, the computational results suggest that the SNF of the distance Laplacian matrix seems to perform better for determining graphs. Finally, we show that complete graphs and star graphs are determined by the SNF of its distance Laplacian matrix.

This is joint work with Aida Abiad (Eindhoven University of Technology and Ghent University), Kristin Heysse (Macalester College), Libby Taylor (Stanford University) and Marcos C. Vargas (Banco de México).

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Two moments for trees

Lorenzo Ciardo

University of Oxford

The moment of a force $\mathbf{F}$ applied to a point particle having distance $\mathbf{d}$ from a fixed fulcrum is the cross product $\mathbf{d} \times \mathbf{F}$. We consider two graph-theoretic versions of this notion, of different nature: Given a rooted tree T, its *combinatorial* moment μ is given by the sum over each vertex v of the distance of v from the root times the degree of v; its *spectral* moment ρ is the largest eigenvalue of a square matrix encoding the "common distance" from the root of pairs of vertices in T. The features of both these parameters resemble those of their physical counterpart. Therefore, they share a similar behaviour with respect to elementary constructions on trees. This allows us to show that μ is essentially an upper bound for ρ, and the ratio μ/ρ is at most linear in the order of T; specific classes of trees having a fractal structure allow to conclude that μ/ρ is in fact unbounded in general.

Interestingly, both μ and ρ are closely linked to connectivity notions for graphs – Kemeny's constant κ and algebraic connectivity α, respectively. As a consequence, the quantitative comparison between the two moments promises to shed some light on the still shadowy relation between κ and α.

Spectra of Variants of Distance Matrices of Graphs

Leslie Hogben

Iowa State University and American Institute of Mathematics

In the last ten years, variants of the distance matrix of a graph, such as the distance Laplacian, the distance signless Laplacian, and the normalized distance Laplacian matrix of a graph, have been studied. This talk compares and contrasts techniques and results for these four variants of distance matrices. New results are obtained by cross-applying techniques from one variant of the distance matrix to another are presented.

This is joint work with Carolyn Reinhart (Swarthmore).

The distance matrix and its variants for digraphs

Carolyn Reinhart

Swarthmore College

A directed graph, or digraph, is a graph in which edges are replaced by directional arcs. While the distance matrix and its variants are symmetric matrices when defined on graphs, these matrices are not necessarily symmetric on digraphs. Thus, some of the techniques used in the graph case no longer apply. This talk will discuss techniques used to study distance matrices for digraphs and some results they have yielded. New results regarding cospectrality for the distance matrix of digraphs will also be presented.

This is joint work with Leslie Hogben (Iowa State University and AIM).

MS-9: Linear algebra education

Organisers: Anthony Cronin (University College Dublin) and Sepideh Stewart (University of Oklahoma)

Theme: Given the technological advancements of the modern era, the teaching and learning of linear algebra has never been more important for students. This minisymposium aims to draw out the challenges and highlight current practice in linear algebra instruction. The 12 talks from presenters from 5 countries will include topics such as: Technology enhanced learning, What Should We Teach in Elementary Linear Algebra Courses Today, Motivating Undergraduate Spectral Theory with Computer Labs, Training maths support tutors with linear algebra specific skills, Student understanding of proof and rigour in a second course in university linear algebra,among many others.

Analysis of Tutors' Feedback After Responding to Linear Algebra Students' Queries

Anthony Cronin and Sepideh Stewart

University College Dublin and Oklahoma University

Using Mason's (2002) pedagogical tactics, we created a conceptual framework to analyze mathematics tutors' responses to linear algebra students' queries in a mathematics support center (MSC). The aim was to investigate the nature of students' difficulties with concepts in a second linear algebra course emphasizing theories and proof, from the perspective of MSC tutors. We examined tactics employed by these tutors to resolve student difficulties. We analyzed 227 feedback comments from 44 tutors based on their interactions with 82 students over six years. Our findings indicated that the most common areas of difficulty were basis, span (and their connection), in addition to vector space, subspace, and proof. Tutor tactics deployed included: simplifying and complexifying, sense making of definitions and theorems, discussion with students, and providing examples via a variety of representations. In this talk we will discuss implications for linear algebra tutor training and indicate some future work.

Interactionist perspective on negotiation processes of students' different understandings during small group work on linear algebra

Ann Sophie Stuhlmann

Universität Hamburg

Student group work represents a central learning setting within mathematics programs at the university level. In my study, a theoretical perspective on collaboration is adopted in which the differences between students' interpretations of a mathematical concept are seen as an opportunity for individual restructuring processes [1]. This so-called interactionist perspective is applied to student group work on linear algebra. The concepts of linear algebra at the university level are characterized by a versatility of different modes of expression and interpretation [2]. For instance, the concept of the dual space of a vector space, represents the vector space of all linear forms of the vector space into its corresponding field. This kind of nested structure of linear algebra concepts requires a cognitive flexibility that allows switching between the different set levels and adopting different interpretations of the corresponding set elements. For students of linear algebra, the flexible transitions between the different interpretations of linear algebra concepts usually pose a challenge. This study focuses on how students negotiate their different interpretations during group work on linear algebra and how transitions between interpretations might be stimulated or hindered. Video recordings of eight student groups, each working on two different tasks, were sampled. The first task required flexible transition between interpretations of group homomorphisms and the second one transitions between different viewpoints on linear forms in the context of dual spaces. The recordings were analyzed from an interactionist perspective, focusing on interaction situations in which the participating students expressed and negotiated different interpretations of the group homomorphisms resp. linear forms. The analyses show students' difficulties in communicating about linear algebra concepts that can be expressed and interpreted differently.

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Linear combinations of vectors in Inquiry-Oriented Linear Algebra (IOLA)

Michelle Zandieh

Arizona State University

Linear combinations of vectors are ubiquitous across topic areas in a first course in linear algebra. This presentation will report on students' thinking with linear combinations of vectors in a selection of tasks from the Inquiry-Oriented Linear Algebra (IOLA) curriculum development and research projects. The IOLA team leverages Realistic Mathematics Education (RME) heuristics in our research design cycle. In particular, we use experientially real starting points to engage students in activities that build toward fundamental ideas in linear algebra. Linear combinations have been particularly important throughout our curriculum design. The IOLA curriculum takes a vector first approach by starting with travel as a metaphor for linear combinations of vectors. The Magic Carpet Ride (MCR) task sequence focuses on the guided reinvention of span and linear independence by engaging students with tasks about travel vectors including questions about what results are possible given different travel vectors or goal locations.

For this presentation I will discuss student thinking with two extensions of this task that leverage the travel metaphor in new ways. The first of these is a new IOLA task sequence asking students to explore the situation when the travel vectors cannot reach the intended destination and leading to the construction of a least squares solution. The second is our development of two digital games that combine RME with principles of game-based learning to engage students with linear combinations. Students manipulate linear combinations of vectors to maneuver their avatar through a game scenario. The game feedback allows for both overlapping and unique strategies when compared with the MCR task.

This is joint work with Dr. Megan Wawro (Virginia Tech), Dr. Christine Andrews-Larson (Florida State University) and Dr. David Plaxco (Clayton State University). Supported by the National Science Foundation, Grant DUE-1914841. This is joint work with Ashish Amresh (Arizona State University) and Dr. David Plaxco (Clayton State University). Supported by the National Science Foundation, Grant DUE-1712524.

Incorporating Tensors into Linear Algebra Courses

John Sheekey

University College Dublin

Tensors are fundamental mathematical objects that arise in a variety of areas of mathematics and physics. In finite dimensions they can be seen as a natural extension of matrices to higher dimensional arrays, or more generally as multilinear maps and forms. Students often first encounter tensors in courses on differential geometry or applied mathematics. However, there are many interesting and accessible applications of tensors in the realm of linear algebra.

In this talk we will share some ideas for motivating the study of tensors in an advanced linear algebra course. We will discuss classical applications such as the complexity of multiplication in an algebra, as well as more recent applications arising from quantum information theory and post-quantum cryptography.

Students' Perspectives on Proofs in Linear Algebra: Ways of Thinking and Ways of Understanding in the Formal World

Sepideh Stewart and Anthony Cronin

University of Oklahoma and University College Dublin

Many mathematics departments offer a second course in linear algebra. However, research on teaching and learning the topics in second courses are scarce. To help fill this gap in the literature, in this study, we interviewed 18 students taking a second linear algebra course in both the USA and Ireland. The theoretical framework is based on Tall's (2008) formal world of mathematical thinking and Harel's (2008) ways of thinking and ways of understanding. The goal of the study was to gain an understanding of the teaching and learning of linear algebra proofs from students' perspectives.

This work is in collaboration with Tien Tran and Aidan Powers

Student Reasoning about Linear Algebra in Quantum Mechanics

Megan Wawro

Virginia Tech

Linear algebra is central in solving many quantum mechanics problems. Students often utilize mathematical concepts and procedures, mathematize physical constructs in terms of mathematical structures, and interpret mathematical entities in terms of a physical context. In this talk, I summarize findings from two research studies based on interviews with quantum mechanics students. In the first study, students were asked to determine the probabilities with which measuring S_z and S_y would yield $\pm\frac{\hbar}{2}$; results focus on how students' reasoning with orthonormal bases, change of basis, and inner products informed their flexibility in choosing problem-solving approaches. In the second study, students were asked to explain what the equations $A\vec{x} = \lambda\vec{x}$ and $\hat{S}_x \left|+\right\rangle_x = \frac{\hbar}{2} \left|+\right\rangle_x$ meant to them and to compare and contrast how they conceptualized eigentheory in the two situations; results focus on students' nuanced imagery for the eigenequations and highlight instances of synergistic and potentially incompatible interpretations. I hope the research findings spur conversation about the relationship between what is taught in linear algebra courses and quantum mechanics courses, and what experiencing and making sense of both courses might be like for students.

This is supported by the United States National Science Foundation, DUE-1452889.

Application Approach to Teaching Linear Algebra

Amanda Harsy, Michael Smith

Lewis University, Lewis University,

This presentation is a follow-up talk to "Inspiring Linear Algebra Topics Using Image and Data Applications" during which we will share our creation of a first-year applied linear algebra course. This course implements IMAGEMath modules and other activities which connect linear algebra concepts to applications such as computer graphics and sports analytics. We have found that giving students the opportunity to learn accessible modeling techniques used by researchers enhances their experience in their mathematics courses and provides them with a clear application of linear algebra concepts. We organize these activities as 50-minute lab modules in our own classes, and we will also share ways of transforming these exercises into smaller in-class lessons or larger semester-long research projects. These activities have been implemented in a variety of teaching modalities including asynchronous and synchronous online classes. This is joint work with Tom Asaki, Heather Moon, and Marie Snipes.

This is a joint presentation with Amanda Harsy and Michael Smith.

16 Questions and Answers for a Modern first Linear Algebra and Matrix Theory Course

Frank Uhlig

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This is a sequel to my Seattle Joint Math Meeting talk (zoom in April 2022) titled "Taking our First Linear Algebra Course into the Third Millennium", see `http://webhome.auburn.edu/~uhligfd/Talkslides2022`.

Be advised to read or watch the invited 1 hour JMM talk in preparation for these questions as we discuss and try to answer them now at ILAS 2022 in Galway.

This session is book-ended in front by the JMM Linear Algebra Education session in Seattle and in back by the ILAS Education Committee's efforts to create and gather lesson plans, pedagogical advice and problem sets that will deal with Linear Algebra at the elementary and separately at the applied level in light of modern Matrix Theory.

In between the book-ends I like to ask the audience to please share their ideas and their history with Linear Algebra freely so that our community's consciousness levels can be appraised for our future teaching efforts and your understanding of common teaching successes and possible failures.

From beginner to expert, increasing linear algebra fluency and comfort with Python labs.

Emily J. Evans

Brigham Young University

Nine years ago, a new program in applied and computational mathematics was introduced at Brigham Young University that included a set of Python-based computer labs to reinforce our teaching of advanced spectral theory. Based on the success of this program, we have now introduced Python-based labs that span our linear algebra curriculum from the earliest students to graduating seniors. In this talk, I will address not only the topics in which we teach but also some of the logistics including getting buy-in from established faculty. I will also focus on how we introduce, motivate, and teach topics typically not seen until graduate school including the Perron-Frobenius theorem, the spectral mapping theorem, Krylov subspace methods, and the pseudospectrum.

This is joint work with Mark Hughes (BYU), Jeff Humpherys (University of Utah) and Tyler Jarvis (BYU) Supported by the National Mathematics Foundation, DUE-TUES Grant Number 1323785.

Inspiring Linear Algebra Topics Using Image and Data Applications

Heather Moon and Marie Snipes

Washington State University and Kenyon College

In this talk we describe an application-first approach to teaching introductory linear algebra. Students begin with explorations of two imaging applications, radiography and tomography, and image manipulation with heat diffusion, and then proceed to learn about the tools of linear algebra in the context of those applications. Our goal is for the context to create a need for the development of linear algebra concepts and tools. In this talk we showcase a few of the activities we developed for students as part of the IMAGEMath project and we outline how, together, the two applications inspire most of the key topics in a first course in linear algebra.

This is joint work with Tom Asaki (WSU). Supported by the National Science Foundation, Grants DUE-1503929, DUE-1642095, DUE-1503870, and DUE-1503856.

Pedagogy of linear combination and the levels of thinking about linear combination

Günhan Caglayan

New Jersey City University

In the course of my teaching of linear algebra along with my research studies on the pedagogy of linear algebra, I identified the following (not-necessarily exhaustive nor hierarchical) levels of thinking about the notion of linear combination:

[1] Verifying that one of the vectors in the set S is a linear combination of the other vectors when the coefficients to form the linear combination are given

[2] Following a procedure in order to write a given vector as a linear combination of the vectors in the set S when the coefficients to form the linear combination are not given

[3] Declaring a given set S as a spanning set for a vector space

[4] Declaring a given set S as a linearly independent set

[5] Establishing a given set S as a basis for a vector space

[6] Determining whether a given vector is in the column space of the matrix whose columns are made of the vectors in a given set S

[7] Obtaining the coordinate vector representation of a given vector relative to a certain basis S

[8] Obtaining the matrix representation of a given linear transformation relative to a certain basis S

[9] Obtaining the matrix representation of a given linear transformation relative to bases S and S'

[10] Obtaining the diagonal matrix representation of a given linear transformation relative to a certain basis S

[11] Obtaining the matrix representation of a given linear transformation relative to an eigenbasis S

This presentation will focus on these proposed levels of understanding of linear combination in an attempt to possibly revise / reorder them in a progressive manner from the least to the most sophisticated. The possibility of including additional levels of linear combination in the aforementioned list will also be considered.

Matrix zeros of polynomials

Damjan Kobal

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The concepts of polynomials and matrices essentially expand and enhance the elementary arithmetic of numbers. Once introduced, polynomials and matrices open up new interesting mathematical challenges which extend to new fields of mathematical explorations within university mathematics. We present an aspect of a rather elementary exploration of polynomials and matrices, which offers a new perspective and an interesting matrix analog to the concept of a zero of a polynomial. The discussion offers an opportunity for better comprehension of the fundamental concepts of polynomials and matrices. As an application we are lead to the meaningful questions of linear algebra and to an easy proof of otherwise advanced and abstract Cayley - Hamilton theorem.

MS-10: Numerical linear algebra for PDEs

Organiser: Niall Madden (NUI Galway)

Theme: This mini-symposium will feature takes on varied topics broadly related to linear and nonlinear solvers for problems arising from the discretization of PDEs. As such, it will include elements of both theoretical and applied numerical linear algebra.

A boundary-layer preconditioner for singularly perturbed convection diffusion problems

Niall Madden

National University of Ireland, Galway

The numerical analysis of discretizations of singularly perturbed differential equations is an established sub-discipline within the study of the numerical approximation of solutions to differential equations. The motivatation stems the wide range of real-world problems whose solutions exhibit boundary and interior layers, and the challenges posed when trying to solve these problems numerically.

Consequently, much is known about how to accurately and stably discretize such equations in order to properly resolve the layer structure present in their continuum solutions. However, despite being a key step in the numerical simulation process, the study of efficient and accurate solution of the associated linear systems is somewhat neglected (though not entirely, see, e.g., [1, 2, 4]).

In this talk, we discuss problems associated with the application of direct solvers to these discretizations. We then propose a preconditioning strategy that is tuned to the matrix structure induced by using layer-adapted meshes for convection-diffusion equations, proving a strong condition-number bound on the preconditioned system in one spatial dimension, and a weaker bound in two spatial dimensions. Numerical results confirm the efficiency of the resulting preconditioners in one and two dimensions, with time-to-solution of less than one second for representative problems on 1024×1024 meshes and up to $40\times$ speedup over standard sparse direct solvers.

This talk is based on [3]; see also `https://arxiv.org/abs/2108.13468`.

This is joint work with Scott P. MacLachlan (Memorial University) and Thái Anh Nhan (Holy Names University).

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A scalable and robust vertex-star relaxation for high-order FEM

Patrick E. Farrell

University of Oxford

High-order finite element methods (FEM) offer numerous advantages. They are especially attractive on modern supercomputers, due to their arithmetic intensity and rapid convergence for smooth solutions. However, all aspects of a code must change at high-order, from the choice of basis functions to matrix-free assembly strategies and on to postprocessing and visualisation. A particularly important challenge is to develop preconditioners that operate matrix-free, without ever accessing even the local tensor for a single cell.

One promising strategy for preconditioners for high-order FEM is the use of p-multigrid. Pavarino proved that the two-level method with vertex patch relaxation for the high-degree problem and a low-order coarse space gives a solver that is robust in polynomial degree for symmetric and coercive problems [1]. However, for very high polynomial degree it is not feasible to assemble or factorize the matrices for each vertex patch, since they are dense.

In this work we introduce a direct solver for separable vertex patch problems that scales to very high polynomial degree on tensor product cells. The solver constructs a carefully-chosen tensor product basis that diagonalizes the blocks in the stiffness matrix for the internal degrees of freedom of each individual cell. As a result, the non-zero structure of the cell matrices is that of the graph connecting internal degrees of freedom to their projection onto the facets. In the new basis, the patch problem is as sparse as a low-order finite difference discretization, while having a sparser Cholesky factorization. We can thus afford to assemble and factorize the matrices for the vertex-patch problems, even for very high polynomial degree. In turn, this enables the use of fast p-multigrid solvers. In the non-separable case, the method can be applied as a preconditioner by approximating the problem with a separable surrogate.

We demonstrate the approach by solving the Poisson equation and a $H(\mathrm{div})$-conforming interior penalty discretization of linear elasticity in two dimensions at polynomial degree $p = 31$ and in three dimensions at $p = 15$.

This is joint work with Pablo D. Brubeck (Oxford). Supported by a Mathematical Institute departmental scholarship, and the Engineering and Physical Sciences Research Council, grants EP/R029423/1 and EP/W026163/1.

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Flexible infinite GMRES for parameterized linear systems

Siobhán Correnty

KTH Royal Institute of Technology

We seek the numerical solution to the large sparse linear system

$$A(\mu)x(\mu) = b, \tag{2}$$

where $\mu \in \mathbb{C}$, $A(\mu) \in \mathbb{C}^{n\times n}$ nonsingular, analytic and nonlinear in μ, and $b \in \mathbb{C}^n$. Under these assumptions, the matrix $A(\mu)$ can be expressed locally by an infinite Taylor series expansion centered around origin, i.e.,

$$A(\mu) = \sum_{\ell=0}^{\infty} A_\ell \mu^\ell, \qquad A_\ell := A^{(\ell)}(0)\frac{1}{\ell!} \in \mathbb{C}^{n\times n}. \tag{3}$$

In our setting, we assume further that the Taylor coefficients in (3) do not vanish after a certain degree, and many of the derivatives of $A(\mu)$ are computationally available. The method proposed here efficiently approximates the solution to (2) for many values of the parameter μ simultaneously. This novel approach offers a significant reduction in complexity over the prior work [1].

The nonlinear dependence on the parameter μ in (2) is addressed with a technique called companion linearization, commonly used in the study of polynomial eigenvalue problems. The arising system, linear in the parameter μ, is approximated within a flexible right-preconditioned GMRES framework. The basis matrix for the Krylov subspace is built just once using the infinite Arnoldi method [2], a process independent of the truncation parameter m. As this process can be carried out in a finite number of operations, we, in theory, take $m \to \infty$ while constructing the basis matrix.

The preconditioner is applied almost exactly when the residual of the outer method is large, and with decreasing accuracy as the residual is reduced, as proposed in [3]. In practice, the level of accuracy can be relaxed dramatically without degrading convergence.

We analyze our method in a way which is analogous to the standard convergence theory for the method GMRES for linear systems. The competitiveness of our method is illustrated with large-scale problems arising from a finite element discretization of a Helmholtz equation with parameterized material coefficient.

This is joint work with Elias Jarlebring (KTH Royal Institute of Technology) and Kirk M. Soodhalter (Trinity College Dublin). This work was funded by The Swedish research council (VR).

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Analysis of block GMRES using a *-algebra-based approach

Kirk M. Soodhalter

Trinity College Dublin

We discuss the challenges of extending convergence results of classical Krylov subspace methods to their block counterparts and propose a new approach to this analysis. Block KSMs such as block GMRES are generalizations of classical KSMs, and are meant to iteratively solve linear systems with multiple right-hand sides (a.k.a. a block right-hand side) all-at-once rather than individually. However, this all-at-once approach has made analysis of these methods more difficult than for classical KSMs because of the interaction of the different right-hand sides. We have proposed an approach built on interpreting the coefficient matrix and block right-hand side as being a matrix and vector over a *-algebra of square matrices. This allows us to sequester the interactions between the right-hand sides into the elements of the *-algebra and (in the case of GMRES) extend some classical GMRES convergence results to the block setting. We then discuss some challenges which remain and some ideas for how to proceed.

This is joint work with Marie Kubiínová from Czech Academy of Sciences, Institute of Geonics, Ostrava, Czech Republic (formerly)

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Preconditioned iterative methods for multiple saddle-point systems arising from PDE-constrained optimization

John W. Pearson

University of Edinburgh

Optimization problems subject to PDE constraints form a mathematical tool that can be applied to a wide range of scientific processes, including fluid flow control, medical imaging, biological and chemical processes, and many others. These problems involve minimizing some function arising from a physical objective, while obeying a system of PDEs which describe the process. Of key interest is the numerical solution of the discretized linear systems arising from such problems, and in this talk we focus on preconditioned iterative methods for these systems.

In particular, we describe recent advances in the preconditioning of multiple saddle-point systems, specifically positive definite preconditioners which can be applied within MINRES, which may find considerable utility for solving these optimization problems as well as other applications. We discuss an inexact active-set method for large-scale nonlinear PDE-constrained optimization problems, coupled with block diagonal and block triangular preconditioners for multiple saddle-point systems which utilize suitable approximations for the relevant Schur complements.

Further, we discuss an alternative structure of a preconditioner for multiple saddle-point systems, which may be applied within the MINRES algorithm and lead to a guaranteed convergence rate, and often demonstrates superior convergence as opposed to widely-used block diagonal preconditioners.

This is joint work with Andreas Potschka (TU Clausthal), with associated papers available at [1, 2].

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A recycling preconditioning method for crack propagation problems

Xiao-Chuan Cai

University of Macau

In this talk, we discuss a recycling preconditioning method with auxiliary tip subspace for solving a sequence of highly ill-conditioned linear systems of equations of different sizes arising from elastic crack propagation problems discretized by an extended finite element method. The preconditioned linear systems are solved by a Krylov subspace method using a non-trivial initial guess constructed with a modification of an approximate solution around the crack tips. The strategy accelerates the convergence remarkably. Numerical experiments demonstrate the efficiency of the proposed algorithm applied to problems with several types of cracks.

This is a joint work with X. Chen.

Preconditioning the Stage Equations of Implicit Runge Kutta Methods

Michal Outrata

University of Geneva

When using implicit Runge-Kutta methods for solving parabolic PDEs, solving the stage equations is often the computational bottleneck, as the dimension of the stage equations

$$M\mathbf{k} = \mathbf{b}$$

for an s-stage Runge-Kutta method becomes sn where the spatial discretization dimension n can be very large. Hence the solution process often requires the use of iterative solvers, whose convergence can be less than satisfactory. Moreover, due to the structure of the stage equations, the matrix M does not necessarily inherit any of the preferable properties of the spatial operator, making GMRES the go-to solver and hence there is a need for a preconditioner. Recently in [2] and also [1] a new block preconditioner was proposed and numerically tested with promising results.

Using spectral analysis and the particular structure of M, we study the properties of this class of preconditioners, focusing on the eigen properties of the preconditioned system, and we obtain interesting results for the eigenvalues of the preconditioned system for a general Butcher matrix. In particular, for low number of stages, i.e., $s = 2, 3$, we obtain explicit formulas for the eigen properties of the preconditioned system and for general s we can explain and predict the characteristic features of the spectrum of the preconditioned system observed in [1]. As the eigenvalues alone are known to *not* be sufficient to predict the GMRES convergence behavior in general, we also focus on the eigenvectors, which altogether allows us to give descriptive bounds of the GMRES convergence behavior for the preconditioned system.

We then numerically optimize the Butcher tableau for the performance of the entire solution process, rather than only the order of convergence of the Runge-Kutta method. To do so requires to carefully balance the numerical stability of the Runge-Kutta method, its order of convergence, and also the convergence of the iterative solver for the stage equations. We illustrate our result on test problems with an advection-diffusion spatial operator and then outline possible generalizations.

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Provable convergence rate for asynchronous methods via randomize linear algebra

Daniel B. Szyld

Temple University

Asynchronous methods refer to parallel iterative procedures where each process performs its task without waiting for other processes to be completed, i.e., with whatever information it has locally available and with no synchronizations with other processes. For the numerical solution of a general partial differential equation on a domain, Schwarz iterative methods use a decomposition of the domain into two or more (usually overlapping) subdomains. In essence one is introducing new artificial boundary conditions Thus each process corresponds to a local solve with boundary conditions from the values in the neighboring subdomains.

Using this method as a solver, avoids the pitfall of synchronization required by the inner products in Krylov subspace methods. A scalable method results with either optimized Schwarz or when a coarse grid is added. Numerical results are presented on large three-dimensional problems illustrating the efficiency of asynchronous parallel implementations.

Most theorems show convergence of the asynchronous methods, but not a rate of convergence. In this talk, using the concepts of randomized linear algebra, we present provable convergence rate for the methods for a class of nonsymmetric linear systems.

Matrix equation techniques for certain evolutionary partial differential equations

Davide Palitta

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In this talk we show how the linear system stemming from the all-at-once approach for certain evolutionary partial differential equations (PDEs) can be recast in terms of a Sylvester matrix equation which naturally encodes the separability of the time and space derivatives.

Combining appropriate projection techniques for the space operator together with a full exploitation of the structure of the discrete time derivative, we are able to efficiently solve problems with a tremendous number of degrees of freedom while maintaining a low storage demand in the solution process.

Such a scheme can be easily adapted to solve many different time-dependent PDEs and several numerical results are shown to illustrate the potential of our novel approach.

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Extrapolation methods as nonlinear Krylov methods

Conor McCoid

University of Geneva

Krylov methods are commonplace for solving of linear problems. Their use for nonlinear problems requires generalizing them. In linear examples some extrapolation methods have been shown to be equivalent to Krylov subspace methods. Since extrapolation methods can be applied to nonlinear problems, we can view these methods as nonlinear Krylov methods. To show the broad class of equivalences between these methods and others, we build each from their ancestral root-finding method, the multisecant equations, which are an extension of the secant equations to higher dimensions.

This work was completed under the supervision of Prof. Martin J. Gander (Geneva). Supported by the Swiss National Science Foundation.

A Domain Decomposition based preconditioner for Discretised Integral equations in two dimensions

V A Kandappan

Indian Institute of Technology, Madras

In this talk, we present a new preconditioner for dense linear systems arising from discretised integral equations in two dimensions. The developed preconditioner combines the traditional domain decomposition technique with hierarchical matrix representations, in particular the HODLR2D [1]. We apply this preconditioner to improve the conditioning of the system and thereby accelerate the convergence of the iterative solver. We present the preconditioner's performance through numerical experiments on dense linear systems from discretised integral equations in two dimensions. We also compare the performance of the developed new preconditioner with a block diagonal preconditioner.

This is joint work with Sivaram Ambikasaran (Indian Institute of Technology Madras)

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MS-11: The Research and Legacy of Richard A. Brualdi

Organisers: Adam Berliner (St Olaf College), Louis Deaett (Quinnipiac University) and Seth Meyer (St Norbert College)

Theme: Richard Brualdi's career has spanned (no pun intended) nearly six decades. He is not only a prolific researcher and contributor to the linear algebra, graph theory and combinatorics communities, but he also advised 37 Ph.D. students, the most ever for a mathematician at the University of Wisconsin – Madison. This mini-symposium features topics related to and/or inspired by Richard's impressive work.

Richard, Matrices and Polyhedra

Geir Dahl

University of Oslo

It is a pleasure to participate in this session where we honor the work of Richard (A. Brualdi). Richard is, and has been, a central person in our community, and his research in combinatorics and matrix theory is highly acknowledged. Personally I have had the pleasure of knowing and collaborating with Richard for more than 20 years. In this talk I will briefly mention a few topics that have been central in Richard's research, and how they might be connected. The talk will be informal, non-technical and focus on Richard and his mathematics, and our collaboration.

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Loopy 2-graphs

Seth A. Meyer

St. Norbert College

Given a non-increasing sequence $D = (d_1, \ldots, d_n)$ of non-negative integers, we can ask if there exists a symmetric, zero trace, (0,1)-matrix which has row sums equal to the corresponding entries in D. This is equivalent to asking whether or not a simple graph exists with this vertex degree sequence prescribed by D, and can be determined by using the well-known Erdös-Gallai Theorem. When the zero trace condition is relaxed, the natural perspective in the matrix formulation is to consider symmetric (0,1)-matrices with unrestricted trace and prescribed row sums, so that diagonal entries which are 1 – usually thought of as loops in the graph – count 1 towards the vertex degree. However, loops are often considered to contribute 2 towards the degree of the vertex when working in a graph theory context. This is slightly awkward in the matrix, as now off-diagonal entries can be 0 or 1, but diagonal entries can be 0 or 2. However, symmetric matrices with prescribed row sums, zero trace, and entries in $\{0, 1, 2\}$ have been studied and the corresponding existence result to Erdös-Gallai was given by Chungphasian. These can be thought of as graphs without loops where multiedges of multiplicity up to 2 are allowed. Now loops which count 2 are more natural, and we get the class of symmetric matrices with entries in $\{0, 1, 2\}$ but which cannot have '1's on the main diagonal. This talk will explore this class of matrices and present some preliminary results, including necessary and sufficient conditions on the degree sequence for the existence of a matrix in this class and an algorithm which constructs such matrices.

This is joint work with Richard Brualdi (University of Wisconsin).

On the independence of near-vector spaces and their matroids

Karin-Therese Howell, Nancy Ann Neudauer

Stellenbosch University & Pacific University

In *Lineare Algebra über Fastkörpern,* the concept of a vector space, that is, a linear space, is generalized to a structure comprising a bit more non-linearity, the so-called near-vector space by André. Every vector space is a near-vector space, and there are two types of linear independence when one constructs near-vector spaces from finite Dickson nearfields. At the *Workshop for African Women in Discrete Mathematics* in January 2018, Howell asked if anyone thought her work in near-vector spaces has a connection to matroids.

We share some results of this investigation, introducing the matroids of the near-vector spaces as defined by André, where the lack of linearity is as a result of one distributive law not holding in general. André orginally defined independence inside the quasi-kernel, the generating set of a near-vector space. Once we move outside the quasi-kernel, it quickly becomes apparent that some strange things can happen, very unlike what we know for vector spaces, as we will see.

We characterise the independence of near-vector spaces constructed using copies of finite fields. We show that for regular near-vector spaces of this nature, independence is equivalent to the notion of independence in the associated vector space. A highlight is proving that for the construction where the number of maximal regular subspaces coincides with the dimension, any element outside of the quasi-kernel can generate the entire space. We completely characterise independence for this space. We define matroids for finite field constructions and those using copies of a proper finite near-field.

This joint work is supported by the National Research Fund (South Africa) (Grant number: 96056), a Simons Foundation Mathematics Collaboration grant, the Thomas and Joyce Holce endowed professorship, and the African Institute of Mathematical Sciences (AIMS South Africa) Research Centre.

Richard's mathematical legacy that influenced Korea

Gi-Sang Cheon

Sungkyunkwan University, Suwon 16419, South Korea

The field of combinatorics and matrix theory in Korea did not become so active until Richard's Ph.D. students returned to Korea in mid-1980 from Wisconsin. Afterwards, many people have become interested in the interaction between combinatorics and matrix theory due to their active research activities and education at graduate school. As a result, combinatorial matrix theory has become one of the active research fields in Korea. Moreover, the International Conference on Combinatorial Matrix Theory (Co-Chair: Richard Brualdi, Suk-Geun Hwang) was first held in Pohang on January 14-17, 2002 with the support of the National Research Foundation of Korea. A large number of prominent scholars in this field were invited and the event was held successfully with about 100 participants. Since then, the research field has become more diverse, and the 19th ILAS Meeting was held in Seoul on August 6-9, 2014 as a satellite conference for ICM 2014. The purpose of the meeting was to promote research interaction in all aspects of linear algebra and its applications. In addition, the International Conference on Matrix theory and Applications to Combinatorics, Optimization, and Data analysis was held in Jeju on May 23-27, 2019. Richard gladly accepted the invitation to speak at the plenary session, and the meeting was even more successful.

We are always grateful to Richard for his direct and indirect assistance in these research activities in Korea. In this talk, we look back on how he influenced the research and education of combinatorics and matrix theory in Korea.

On the spectrum of graduate research projects with Richard Brualdi

Michael William Schroeder (#35)

Marshall University

As a graduate student, I was a coauthor with Richard and his other graduate students at the time on six different papers. While these publications were all in matrix theory, their topics and methods were largely distinct, both from each other and each graduate student's dissertation. I attribute my modest success as a collaborative researcher from these rewarding experiences.

In this talk, I give brief mention to each of these papers, as well as some further results that have stemmed from these publications. This is joint work with a variety of Brualdites, particularly #33, #34, #36, #37 and, of course, #0.

Permanents of t-triangular $(0,1)$-matrices

John Goldwasser

West Virginia University

Let A be a square matrix and t a positive integer. We say A is t-triangular if there exist permutation matrices P and Q such that $PAQ = B = [b_{ij}] = 0$ whenever $j \geq i + t$. We ask for which positive integers the following statement is true: If A is any square matrix with nonnegative integral entries such that $0 < \operatorname{per} A < (t+1)!$, then A is t-triangular. If $t = 1$, the statement reduces to a theorem of Brualdi. I will show the statement is true for $t = 2$ and 3, but false for $t = 6$.

Determinants:Digraphs::Pfaffians:Matchings

Jennifer J. Quinn

University of Washington Tacoma

Determinants have a beautiful combinatorial interpretation as non-intersecting path systems on acyclic digraphs due to Lindström, Gessel, and Viennot[1, 2] that lead to intuitive proofs of determinant identities. Since Pfaffians, defined on skew-symmetric matrices, are essentially the square root of the determinant, can the same be said for Pfaffian identities? This talk explores combinatorial interpretations of Pfaffians, determinants, and the connections between them. It showcases sign reversing involutions, a powerful and often underappreciated combinatorial method.

This is joint work with Naiomi Cameron (Spelman College).

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Pattern-Avoiding Permutation Matrices

Richard A. Brualdi

University of Wisconsin - Madison

Permutation matrices appear throughout combinatorial matrix theory. Permutation patterns form a rich part of the combinatorial theory of permutations. In [1] we introduced a new saturation function for $m \times n$ $(0,1)$-matrices: A is *saturating* for a $(0,1)$-pattern P if A does not contain the pattern P (A *avoids* P) and A is maximal with respect to this property (no 0 can be changed to a 1). We proved, among other things, that the saturation function for the pattern $P = I_k$ (so $12 \cdots k$ as a permutation) equals $(k-1)(m+n-(k-1))$, and that if A has fewer 1's, some 0 of A can be replaced with a 1 so that A still avoids I_k. A similar result is obtained for the permutation pattern 312 (the only other essentially different permutation pattern with $k=3$).

In [2] we are motivated by some old work of Fulkerson that has some connection with the famous Frobenius-Kőnig theorem, namely *blocking* permutation matrices in minimal and minimum ways. In the F-K situation, every $r \times s$ submatrix of an $n \times n$ $(0,1)$-matrix with $r+s = n+1$ blocks all $n \times n$ permutation matrices; in particular every row and column, so n positions. For patterns of size $k > 3$, we show that the only blockers of size n are the rows and columns (so, in fact they block all n-permutations). If $k = 3$, a minimal blocker must be of size n, but need not be a row or column.

In [3] we investigate continuous analogues of some of these investigations, namely convex hulls of pattern-avoiding permutation matrices, a generalization of the polytopes of doubly stochastic matrices.

This talk is based on continuing joint work with Lei Cao of Nova Southeastern University, Florida [1, 2, 3].

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MS-12: Matrix positivity: theory and applications

Organisers: Alexander Belton (Lancaster University) and Dominique Guillot (University of Delaware)

Theme: Matrix positivity, or positive semidefiniteness, is one of the most wide-reaching concepts in mathematics, old and new. Positivity of a matrix is as natural as positivity of mass in statics or positivity of a probability distribution. It is a notion which has attracted the attention of many great minds. Yet, after at least two centuries of research, positive matrices still hide enigmas and raise challenges for the working mathematician. The vitality of matrix positivity comes from its breadth, having many theoretical facets and also deep links to mathematical modelling. The speakers in this minisymposium work on various aspects of this subject, both pure and applied.

Riordan arrays: structure and positivity

Paul Barry

SETU, Ireland

Riordan arrays arise from the matrix representation of the Riordan group, whose elements are pairs of formal power series. These arrays thus have an algebraic structure and a matrix structure. We examine aspects of these structures, and show how they are inter-linked, before turning to look at positivity results for Riordan arrays.

Positivity preservers forbidden to operate on diagonal blocks

Prateek Kumar Vishwakarma

University of Regina, Canada

The question of which functions acting entrywise preserve positive semidefiniteness has a long history, beginning with the Schur product theorem [4], which implies that absolutely monotonic functions (i.e., power series with nonnegative coefficients) preserve positivity on matrices of all dimensions. A famous result of Schoenberg and of Rudin [2, 3] shows the converse: there are no other such functions.

Motivated by modern applications, Guillot and Rajaratnam [1] classified the entrywise positivity preservers in all dimensions, which act only on the off-diagonal entries. These two results are at "opposite ends", and in both cases the preservers have to be absolutely monotonic.

We complete in [5] the classification of positivity preservers that act entrywise except on specified "diagonal/principal blocks", in every case other than the two above. (In fact we achieve this in a more general framework.) This yields the first examples of dimension-free entrywise positivity preservers - with certain forbidden principal blocks - that are not absolutely monotonic.

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An Atomic Viewpoint of the Totally Positive Completion Problem

Daniel Carter

Princeton University

We present two complementary techniques called catalysis and inhibition which allow one to determine if a given pattern is TP completable or TP non-completable, respectively. Empirically, these techniques require considering only one unspecified entry at a time in a vast majority of cases, which makes these techniques ripe for automation and a powerful framework for future work in the TP completion problem. With small modifications, these techniques are also applicable to the TN completion problem.

We provide two major applications. First, we characterize all 4-by-4 patterns by completability. There are a total of 78 new obstructions of this size, six times as many as the 3-by-n case for all n combined. Second, we provide a characterization of the so-called 1-variable obstructions in the TN case, which includes as a corollary a characterization of patterns with a single unspecified entry. This also provides a novel partial result towards proving the conjecture that all TN-completable patterns are TP-completable.

This is joint work with Charles Johnson (College of William and Mary). Supported by the National Mathematics Foundation, Grant DMS-0751964.

Maximizing the number of positive eigenvalues of an LCM matrix

Mika Mattila

Tampere University

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers with $x_i \leq x_j \Rightarrow i \leq j$. The GCD matrix (S) of the set S is the $n \times n$ matrix with $\gcd(x_i, xj)$ as its ij entry. Similarly, the LCM matrix $[S]$ of the set S has $\mathrm{lcm}(x_i, x_j)$ as its ij entry. Both of these matrices were originally defined by H. J. S. Smith in his seminal paper from the year 1876.

During the last 30 years both GCD and LCM matrices (as well as their various generalizations) have been investigated extensively in the literature. Although the entries of both of the matrices are all positive integers, the properties of these matrix types differ greatly from each other. For example, the GCD matrix (S) is positive definite for any set S whereas every nontrivial LCM matrix $[S]$ is indefinite and may be even singular. Still, the number of positive and negative eigenvalues of the matrix $[S]$ varies a lot depending on the actual elements of the set S. This gives raise to a new question: how to construct an LCM matrix that has as many positive eigenvalues as possible?

It this talk we shall focus solely on the cases when the set S is GCD closed, because in this situation the poset-theoretic semilattice structure of $(S, |)$ often alone determines the inertia of the LCM matrix $[S]$ completely. This may be a bit surprising, since one could expect the exact values of the elements $x_i \in S$ to play a bigger role in this. Nevertheless, our method makes it possible to give examples of matrices $[S]$ for which only a "small portion" of the eigenvalues are negative.

The presentation is based on the content of the Section 5 of the article [1].

This is joint work with Pentti Haukkanen (Tampere University) and Jori Mäntysalo (Tampere University).

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Completing an Operator Matrix and the Free Joint Numerical Radius

Hugo J. Woerdeman

Drexel University

Ando's [1] classical characterization of the unit ball in the numerical radius norm was generalized by Farenick, Kavruk and Paulsen [2] using the free joint numerical radius of a tuple of Hilbert space operators $(X_1, \ldots, X_m)$. In particular, the characterization leads to a positive definite completion problem. In this paper we study various aspects of Ando's result in this generalized setting. Among other things, this leads to the study of finding a positive definite solution L to the equation

$$L = I + \sum_{j=1}^{m} \left[\left(L^{\frac{1}{2}} X_j^* L X_j L^{\frac{1}{2}} + \frac{1}{4} I \right)^{\frac{1}{2}} + \left(L^{\frac{1}{2}} X_j L X_j^* L^{\frac{1}{2}} + \frac{1}{4} I \right)^{\frac{1}{2}} \right],$$

which may be viewed as a fixed point equation. Once such a fixed point is identified, the desired positive definite matrix completion is easily identified. Along the way we also derive new formulas for the joint numerical radius when the tuple consists of generalized permutations. Finally, we present some open problems.

This is joint work with Kennett L. Dela Rosa (University of the Philippines Diliman). Supported by Simons Foundation grant 355645 and National Science Foundation grant DMS 2000037

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Coefficientwise total positivity of some matrices defined by linear recurrences

Tomack Gilmore

Lancaster University

In this talk I will present some recent results and conjectures [2] concerning the coefficientwise total positivity of a lower-triangular matrix (denoted $T(a, c, d, e, f, g)$) with polynomial entries in six indeterminates that satisfy a three-term linear recurrence. The matrix $T(a, c, d, e, f, g)$ is of particular interest since it includes, as special cases, a number of combinatorially significant integer matrices, two of which are: the Eulerian triangle [3, A008292] (the matrix whose (n, k)-entry counts permutations of $[n]$ with k descents); and the reversed Stirling subset triangle [3, A008278] (the matrix whose (n, k)-entry counts partitions of the set $[n] = \{1, 2, \ldots, n\}$ into $n - k$ non-empty blocks). The former was conjectured to be totally positive over a quarter of a century ago by Brenti [1] and motivated our subsequent research on this topic, while the latter can be shown to be totally positive by specialising our results.

This is joint work with Xi Chen, Bishal Deb, Alexander Dyachenko, and Alan D. Sokal, and was supported in part by the U.K. Engineering and Physical Sciences Research Council grant EP/N025636/1, a fellowship from the China Scholarship Council, and a fellowship from the Deutsche Forschungsgemeinschaft.

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Free functions preserving certain partial orders of operators

Miklós Pálfia

Corvinus University of Budapest and University of Szeged

Recently free analysis has been a very active topic of study in operator and function theory. In particular free functions that preserve partial orders of operators have been studied by a number of authors, in connection to Loewner's theorem. Also operator concave free functions naturally get into the picture as we study the positive definite order preserving free functions. We will go through recent results of the field, and we will cover some recent works on analytic lifts and extension of operator monotone and concave functions to the domain matrix convex hull of their domains. This is related to some conjectures in the field, for instance McCarthy's conjecture. If time permits, we will cover another recent joint work with M. Gaál solving Blecher's problem on characterizing real positive definite order preserving functions.

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MS-13: Rigidity and matrix completion

Organisers: James Cruickshank (NUI Galway) and Derek Kitson (MIC Thurles)

Theme: This minisymposium will focus on geometric rigidity theory and its connections with low rank matrix completion problems.

Graph rigidity in cylindrical spaces

Derek Kitson

Mary Immaculate College, Thurles

We will report on recent progress in characterising minimally rigid graphs for normed spaces of dimension ≥ 3. In particular, we will present combinatorial characterisations of minimal rigidity for the cylindrical spaces $(\mathbb{R}^3, \|\cdot\|_{p,\infty})$ where $\|(x,y,z)\|_{p,\infty} = \max\left\{(|x|^p + |y|^p)^{\frac{1}{p}}, |z|\right\}$ and $p \in (1,\infty)$. As a corollary, we will show that doubly braced sphere triangulations are minimally rigid graphs in the case $p = 2$.

This is joint work with Sean Dewar (JKU, Linz).

When is a rod configuration infinitesimally rigid?

Signe Lundqvist

Umeå University

A rod configuration is a realisation of a hypergraph as points and straight lines in the plane, where the lines behave as rigid bodies. Tay and Whiteley conjectured that the infinitesimal rigidity of rod configurations realising 2-regular hypergraphs depends only on the generic rigidity of body-and-joint frameworks realising the same hypergraph [3]. This conjecture is known as the molecular conjecture because of its applications to molecular chemistry. Jackson and Jordán proved the molecular conjecture in the plane, and Katoh and Tanigawa proved it in arbitrary dimension [1, 2]. Earlier, Whiteley proved a version of the molecular conjecture for hypergraphs of arbitrary that can be realised as independent body-and-joint frameworks in the plane [4].

In this talk, we will see that the infinitesimal rigidity of a sufficiently generic rod configuration realising an arbitrary hypergraph depends only on the generic rigidity of an associated graph, which we call a cone graph. This can be seen as a generalisation of Whiteley's version of the molecular conjecture to arbitrary hypergraphs.

This is joint work with Klara Stokes (Umeå University) and Lars-Daniel Öhman (Umeå University). Supported by the Knut and Alice Wallenberg Foundation, Grant 2020.0001 and 2020.0007.

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Global Rigidity of Frameworks in Non-Euclidean Normed Planes

John Hewetson

Lancaster University

A framework (G, p) is an ordered pair where G is a graph and p maps the vertices of G to some normed space. In the 1990s, Hendrickson [1] gave necessary conditions for a generic framework to be globally rigid in d-dimensional Euclidean space. Connelly proved that Hendrickson's conditions are insufficient when $d \geq 3$, but in 2005 they were shown to be sufficient when $d = 2$. This result combined work by Connelly [2] with a construction of a family of graphs by Jackson and Jordán [3]. More recently, attention has turned to considering frameworks realised in non-Euclidean normed spaces. In this talk we present our characterisation of globally rigid frameworks in analytic (non-Euclidean) normed planes. As in the Euclidean setting, our proof makes use of a relationship between global rigidity of a given framework and the connectivity of a matroid defined on the underlying graph.

This is joint work with Sean Dewar (RICAM) and Tony Nixon (Lancaster).

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Global Rigidity for Line Constrained Frameworks

James Cruickshank

NUI Galway

We will consider the rigidity properties of bar-joint frameworks whose vertices are constrained to lie on a given set of lines in $\mathbb{R}^d$. In particular we will give a necessary and sufficient conditions for a graph to be generically globally rigid in this context, extending previous results of Guler, Jackson and Nixon.

This is joint work with Fatemeh Mohammadi (Ghent), Harshit Motwani (Ghent), Tony Nixon (Lancaster) and Shin-ichi Tanigawa (Tokyo)

A Characterization of Graphs of Super Stable Tensegrities

Shin-ichi Tanigawa

University of Tokyo

Tensegrities are pin-jointed structures made from struts and cables. Super stability introduced by Connelly [1] is one of the widely used sufficient conditions for the (global) rigidity of tensegrities. In this talk, I will give a characterization of graphs that can be realized as super stable tensegrities.

This is joint work with Ryoshun Oba (University of Tokyo). Supported by JST PRESTO Grant Number JPMJPR2126.

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The number of realisations of a minimally rigid graph in various geometries

Sean Dewar

Johann Radon Institute (RICAM)

Given a minimally d-rigid graph G, we define $c_d(G)$ to be the number of d-dimensional realisations of G, and $c_d^*(G)$ to be the number of realisations of G on the d-dimensional sphere. It was computed by Gallet, Grasegger and Schicho that for any Laman graph G with 10 vertices or less, the inequality $c_2(G) \leq c_2^*(G)$ holds; furthermore, this inequality is strict for some, but not all, Laman graphs. In recent ongoing research, we have proven that $c_d(G) \leq c_d^*(G)$ holds for all minimally d-rigid graphs. We obtain this result by first proving that $c_d^*(G) = c_{d+1}(G * o)$, where $G * o$ is the cone of a minimally d-rigid graph G.

This is joint work with Georg Grasegger (Johannes Kepler University). Supported by the Austrian Science Fund (FWF): P31888.

MS-14: History of Linear Algebra

Organisers: Kirk Soodhalter and Jörg Liesen

Theme: "*The evolution of science does not occur in steady growth but in fitful jumps, initiated by sudden flashes of ingenuity which are not different from the manner of artistic creation.*" (C. Lanczos)
This minisymposium will be devoted to some historical flashes of ingenuity that led to fundamental developments in Linear Algebra and its applications, and it will remember some of the founders of the field and their accomplishments.

The invention of character theory (via linear algebra)

Rachel Quinlan

National University of Ireland, Galway

This talk will relate some of the story of the invention of character theory of finite groups, by Dedekind and Frobenius in 1896. Characters are now understood as trace functions of representations, which are homomorphisms from an abstract group to a complex general linear group. They originated however from the efforts of Dedekind to factorize the *group determinant*, a homogeneous polynomial in n variables, where n is the order of the group. By applying very elementary ideas from matrix theory, Dedekind and Frobenius were able to establish most of the fundamental properties of irreducible characters of finite groups. Their ingenious approach is concealed (for good reasons admittedly) in most modern introductions to representation theory and character theory.

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Seventieth anniversary of the conjugate gradient method and what do old papers reveal about our presence

Zdeněk Strakoš

Charles University, Prague

In his lecture *Why Mathematics?* delivered at the Annual Meeting of the Irish Mathematics Association on October 31, 1966, Cornelius Lanczos said: *"The naive optimist who believes in progress and is convinced that today is better than yesterday and in ten years time the world will be infinitely better off than it is today, will come to the conclusion that mathematics (and more generally all the exact sciences) started only about twenty years ago, while the predecessors must have walked around in a kind of limbo of half-digested and improperly conceived ideas."*

This year marks the seventieth anniversary of the paper by Hestenes and Stiefel, which comprehensively described the conjugate gradient method (CG) considered among the most important algorithmic developments of the 20th century. This paper should be studied together with the three closely related papers by Lanczos published within the period 1950-53. It is worth to notice, e.g., also the papers by Karush and Hayes, published in 1952 and 1954, respectively, as well as a couple of other works of several other authors from the same period.

This contribution will examine how the knowledge present in these seminal papers is reflected in the contemporary literature on CG, and what does it show on the status of the current understanding of the deeply rooted mathematical ideas so beautifully presented many decades ago.

The life and the work of André Louis Cholesky

Claude Brezinski

Université de Lille

In this talk, I first describe the life of André Louis Choleky (1873-1918) who was a French army officer specialised in topography and cartography. Then, I analyse his scientific work. In particular, I discuss his well known method for solving a system of linear equations with a symmetric positive definite matrix. I show how this method was forgotten and then came back to light. The other works of Cholesky will also be mentioned.

P. Stein and R.L. Rosenberg

Michela Redivo-Zaglia

University of Padua

In this talk, after reminding the well known theorem of this two researchers, I will speak about their life and works. For the first author it was pretty easy to find information, but for the second one, it was a real puzzle to reconstruct his life. These biographies are included in a forthcoming joint book, written with Claude Brezinski and Gérard Meurant [1] where authors invite the readers to a journey in the history of numerical linear algebra. The second part of the book contains 78 biographies of researchers who contributed significantly to the field of numerical linear algebra, and in the Bibliography there are 3344 references.

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MS-15: Companion Matrix Forms

Organisers: Kevin Vander Meulen (Redeemer University) and Fernando de Terán (Universidad Carlos III de Madrid)

Theme: The Frobenius companion matrix is a well-established classical matrix structure, which has been extensively used over the years, for instance, in the polynomial root-finding problem. Recently there is a renewed interest in companion matrix forms, in part due to the discovery of other companion matrices by Fiedler, and to the interest, in the framework of Nonlinear Eigenvalue Problems, of looking for new classes of linearizations having better numerical features.
This minisymposium will gather researchers interested in the structures of companion matrices and companion linearizations of matrix polynomials and rational matrices.

Error bounds for matrix polynomial eigenvectors

Javier Perez

Department of Mathematical Sciences, University of Montana, USA

The standard approach for computing the eigenvalues and the eigenvectors of a matrix polynomial $P(\lambda) = \lambda^d A_d + \lambda^{d-1} A_{d-1} + \cdots + \lambda A_1 + A_0$ starts by embedding the matrix coefficients A_i into a matrix pencil $\lambda L_1 + L_0$, known as linearization. In this talk, we present novel error bounds for the computed eigenvectors of a matrix polynomial $P(\lambda)$ when the eigenvectors of the polynomial have been recovered from those of a linearization of $P(\lambda)$. We show that, under some linearization-specific conditions, the recovered eigenvectors are almost the exact eigenvectors of a nearby matrix polynomial. These new error bounds can be applied to most of the linearizations introduced in the last decade (companion linearization, $\mathbb{L}_1$ and $\mathbb{L}_2$ linearizations, $\mathbb{DL}$ linearizations, Fiedler linearizations, block Kronecker linearizations, etc). Moreover, we use our theory to show for the first time that the two-linearizations strategy for solving quadratic eigenvalue problems introduced by L. Zeng and Y. Su [1] is backward stable. The theory is illustrated by numerical examples.

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Recovering a perturbation of a matrix polynomial from a perturbation of its companion matrix

Andrii Dmytryshyn

Örebro University

A number of theoretical and computational problems for matrix polynomials are solved by passing to linearizations. Therefore a perturbation theory results for the linearizations need to be related back to matrix polynomials. We present an algorithm that finds which perturbation of matrix coefficients of a matrix polynomial corresponds to a given perturbation its companion matrix [1].

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Applications of companion forms to eigenvalue bounds and scalar polynomials

Aaron Melman

Santa Clara University

We present a simple way to derive companion forms of matrix polynomials, which are lower order matrix polynomials with the same eigenvalues (so-called "ℓ-ifications") as the given matrix polynomial, and show how they can be used to produce eigenvalue bounds. These bounds, which also include non-standard directional ones, can be substantially less computationally demanding when using higher degree companion forms, as opposed to classical linearizations (companion forms of degree one).

As an application to scalar polynomials, we show how companion forms provide a convenient way for the derivation of a polynomial whose (unknown) zeros are powers of those of a given polynomial.

Comparison Theorems for Splittings of M-matrices in block Hessenberg Form

Luca Gemignani

University of Pisa

In this talk we consider the solution of M-matrix linear systems in block Hessenberg form, and we show new comparison results among matrix splittings that hold for this special structure. In particular, we prove that for a lower-Hessenberg M-matrix $\rho(P_{GS}) \geq \rho(P_S) \geq \rho(P_{AGS})$, where $\rho(A)$ denotes the spectral radius of A and P_{GS}, P_S, P_{AGS} are the iteration matrices of the Gauss–Seidel, staircase, and anti-Gauss–Seidel method. This is a result that does not seem to follow from classical comparison results, as these splittings are not directly comparable. Also, it fosters the use of stair partitionings for solving Hessenberg linear systems in parallel.

This is joint work with Federico Poloni (Pisa).

Using the Hessenberg Form of a Sparse Companion Matrix

Kevin Vander Meulen

Redeemer University

A companion matrix can be described as a template for obtaining a matrix with a specified characteristic polynomial. The Frobenius companion matrix is the classic example of such a template. More recently, a broader class of companion matrices were described by Fiedler via a product construction. The Fiedler matrices belong to a larger class of sparse companion matrices that can be characterized by a Hessenberg form. The Hessenberg form enables the calculation of bounds for roots of polynomials. The form also enables the calculation of condition numbers of classes of companion matrices.

Orthogonal iterations on companion-like pencils

Gianna M. Del Corso

University of Pisa

We present a class of fast subspace algorithms based on orthogonal iterations for structured matrices/pencils that can be expressed as small rank perturbations of unitary matrices. The representation of the matrix by means of a new data-sparse factorization –named LFR factorization– using orthogonal Hessenberg matrices is at the core of these algorithms. The factorization can be computed at the cost of $O(nk^2)$ arithmetic operations, where n and k are the sizes of the matrix and the small rank perturbation, respectively. At the same cost from the LFR format we can easily obtain suitable QR and RQ factorizations where the orthogonal factor Q is a product of orthogonal Hessenberg matrices and the upper triangular factor R is again given into the LFR format. The orthogonal iteration reduces to a hopping game where Givens plane rotations are moved from one side to the other side of these two factors. The resulting new algorithms approximate an invariant subspace of size s associated with a set of s leading or trailing eigenvalues using only $O(nks)$ operations per iteration. The number of iterations required to reach an invariant subspace depends linearly on the ratio $|\lambda_{s+1}|/|\lambda_s|$. Numerical experiments confirm the effectiveness of our adaptations.

This is joint work with Roberto Bevilacqua and Luca Gemignani (University of Pisa).

Algebraic Companions

Robert M. Corless

University of Western Ontario

Given companion matrix pencils for polynomials $a(x)$ and $b(x)$, and generalized standard triples for them, one can construct a new companion pencil for $c(x) = xa(x)b(x) + d$ using the smaller pencils as building blocks. Similarly, algebraic linearizations can be built out of smaller linearizations for matrix polynomials.

Working backwards from $c(x)$ is harder, but may be of interest in a search for greater numerical stability. We already know of recursive constructions for the companions for the Mandelbrot polynomial where the eigenvalue condition number is *exponentially* smaller than that of the Frobenius companion matrix. Are there other examples where this approach can be so successful? And what role does "minimal height" play, here?

I gratefully acknowledge the help of Eunice Y.S. Chan, Piers W. Lawrence, and Steven E. Thornton. Discussions with Neil J. Calkin, Laureano Gonzalez-Vega, Nick Higham, J. Rafael Sendra, and Juana Sendra were also very useful. Partially supported by NSERC grant RGPIN-2020-06438, and partially supported by the grant PID2020-113192GB-I00 (Mathematical Visualization: Foundations, Algorithms and Applications) from the Spanish MICINN.

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$\mathbb{DL}(P)$, Bézoutians, and the eigenvalue exclusion theorem for singular matrix polynomials

Vanni Noferini

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Let $P(\lambda)$ be a polynomial matrix. In [3], the vector space $\mathbb{DL}(P)$ of block symmetric potential linearization was defined. In $\mathbb{DL}(P)$, each pencil is associated with a scalar polynomial $v(\lambda)$, called either v-polynomial [3] or ansatz polynomial [4], whose degree is at most the degree of P minus 1. We denote by $\mathbb{DL}(P, v)$ the pencil in $\mathbb{DL}(P)$ associated with the ansatz polynomial $v(\lambda)$. For a regular $P(\lambda)$, pencils in $\mathbb{DL}(P)$ *eigenvalue exclusion theorem* was proved [3]: $\mathbb{DL}(P, v)$ is a strong linearization of $P(\lambda)$ if and only if $v(\lambda)I$ and $P(\lambda)$ do not have any common eigenvalues. Moreover, if $P(\lambda)$ is singular then no pencil in $\mathbb{DL}(P)$ is a linearization [1].

In this talk, we will give arguments based on the connection between $\mathbb{DL}(P)$ and (Lerer-Tismenetsky) generalized Bézoutian matrices [4] and show that even when $P(\lambda)$ is singular an extended eigenvalue exclusion theorem holds. In particular, if $v(\lambda)I$ and $P(\lambda)$ do not have shared eigenvalues, we are able to fully characterize the minimal indices and the partial multiplicities of the corresponding pencil in $\mathbb{DL}(P, v)$. Namely, we show that all the finite and infinite partial multiplicites are the same as in $P(\lambda)$ (the same feature which is true of a strong linearization). Moreover, we show that even if the pencil is not a linearization, it is still possible to recover all the relevant spectral and minimal data of $P(\lambda)$ from those of the pencil, including: minimal indices, partial multiplicities, root polynomials and eigenvectors (defined for singular matrix polynomials in [2]), and minimal bases. In other words, when $P(\lambda)$ is singular and $v(\lambda)$ satisfies the eigenvalue exclusion condition, then $\mathbb{DL}(P, v)$ is an example of a pencil that, albeit not a linerization, offers a recovery of spectral and minimal data which is as attractive as a strong linearization.

This talk is based on joint work with Froilán Dopico (Carlos III Madrid). Supported by the Suomen Akatemia, päätos 331240.

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Linearizations of rational matrices from general representations

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Given a rational matrix $R(\lambda)$, the Rational Eigenvalue Problem (REP) consists of finding scalars λ_0 (eigenvalues) such that there exist nonzero constant vectors x and y (eigenvectors) satisfying

$$R(\lambda_0)x = 0 \quad \text{and} \quad y^T R(\lambda_0) = 0,$$

under the regularity assumption $\det R(\lambda) \not\equiv 0$. The numerical solution of REPs is recently getting a lot of attention from the numerical linear algebra community since REPs appear directly from applications or as approximations to arbitrary nonlinear eigenvalue problems. Rational matrices also appear in linear systems and control theory.

Nowadays, a competitive method for solving REPs is linearization. Linearization transforms the REP into a generalized eigenvalue problem in such a way that the pole and zero information of the corresponding rational matrix is preserved. In this work, we construct a new family of linearizations of rational matrices $R(\lambda)$ written in the general form

$$R(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda),$$

where $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ are polynomial matrices, with $A(\lambda)$ regular. Such representation always exists and are not unique. The new linearizations are constructed from linearizations of the polynomial matrices $D(\lambda)$ and $A(\lambda)$, where each of them can be represented in terms of any polynomial basis. In particular, the block minimal bases linearizations for polynomial matrices in [2] will be our main tool for building linearizations of rational matrices in the sense of [1]. In addition, we show how to recover eigenvectors, when $R(\lambda)$ is regular, and minimal bases and minimal indices, when $R(\lambda)$ is singular, from those of their linearizations in this family. Finally, we show by example how the theory developed in this work can be used for solving (scalar) rational equations of the form

$$\frac{c(\lambda)}{a(\lambda)} = \frac{d(\lambda)}{b(\lambda)},$$

where $a(\lambda)$, $b(\lambda)$, $c(\lambda)$ and $d(\lambda)$ are nonzero scalar polynomials.

This is joint work with Javier Pérez (University of Montana, USA). Work (partially) supported by "Ministerio de Economía, Industria y Competitividad (MINECO)" of Spain and "Fondo Europeo de Desarrollo Regional (FEDER)" of EU through grants MTM2015-65798-P and MTM2017-90682-REDT, and the predoctoral contract BES-2016-076744 of MINECO.

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Primitive Companion Matrices

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In [1] and [2] we studied exponents of primitive companion and primitive symmetric companion matices with enties 0 or 1. A *symmetric companion matrix*, we mean a matrix of the form $A+A^T$, where A is a companion matrix all of whose entries are in $\{0,1\}$ and A^T is the transpose of A. In [2] we found the total number of primitive and imprimitive symmetric companion matrices. We found formulas to compute the exponent of every primitive symmetric companion matrix. Hence the exponent set for the class of primitive symmetric companion matrices is completely characterized. We also obtain the number of primitive symmetric companion matrices with a given exponent for certain cases. *This part is the joint work with Monimala Nej.* In [4] we studied representation of cyclotomic fields and their subfields by circulant matirces and companion matrices.

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Linearizations of matrix polynomials via Rosenbrock polynomial system matrices

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In the seventies, Rosenbrock [3] introduced the concept of a polynomial system matrix $L(\lambda)$ of an arbitrary rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{m\times n}$, where $\mathbb{F}$ is an arbitrary field. Such a system matrix is partitioned in a quadruple $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$ of compatible matrix polynomials

$$L(\lambda) := \begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix}$$

such that its Schur complement with respect to $D(\lambda)$ equals $R(\lambda)$. That is, $R(\lambda) = D(\lambda)+C(\lambda)A(\lambda)^{-1}B(\lambda)$. Since then, the concept of polynomial system matrix has played a key role in linear system theory and control theory. Later, in a fully independent way, I. Gohberg, M. A. Kaashoek, P. Lancaster, and L. Rodman introduced the concepts of linearization [2] and strong linearization [1] of matrix polynomials. The concepts of linearization and strong linearization of matrix polynomials have been widely used in the last two decades, both from theoretical and numerical perspectives, by many authors all over the world, and many explicitly constructible classes of linearizations and strong linearizations have been developed based on them. In this talk, we prove that some of the most important classes of linearizations of matrix polynomials are, modulo block permutations, linear polynomial system matrices whose matrix $A(\lambda)$ is unimodular.

This is joint work with Silvia Marcaida (Universidad del País Vasco UPV/EHU, Spain), María del Carmen Quintana (Aalto University, Finland) and Paul Van Dooren (Université catholique de Louvain, Belgium). This work is part of the "Proyecto de I+D+i PID2019-106362GB-I00 financiado por MCIN/AEI/10.13039/501100011033".

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Non-sparse companion matrices

Louis Deaett

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The familiar Frobenius companion matrix is an $n \times n$ matrix such that $n^2 - n$ of its entries are constant and the remaining n entries have the property that when these are given (in some fixed order) the values $a_1, a_2, \ldots, a_n$, the characteristic polynomial of the resulting matrix is

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n.$$

In 2003, Fiedler [2] introduced a new family of companion matrices meeting the above description. Such matrices have found important applications, naturally in the context of applying linear algebra to problems involving approximating roots of polynomials.

Both the Frobenius and Fiedler companion matrices have the property that each $n \times n$ example has exactly $2n - 1$ nonzero entries, the smallest number possible. Such "sparse" companion matrices were investigated and given a combinatorial characterization in [1].

We consider the notion of a generalized, "non-sparse" companion matrix that results from allowing any number of nonzero entries. (In fact, one problem we explore is that of determining the number of nonzero entries possible in such a matrix.) Some of our results apply to sparse and non-sparse companion matrices alike; e.g., every realization must be non-derogatory. Other results show that some properties known to be true for sparse companion matrices need not hold in the non-sparse case. Finally, we explore what is possible for the combinatorial structure of sparse and non-sparse companion matrices, and highlight some open questions that persist.

This work was done with Jonathan Fischer, Colin Garnett (Black Hills State University), and Kevin Vander Meulen (Redeemer University College) and was supported in part by an NSERC Discovery Grant and an NSERC USRA.

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Non-sparse Companion Hessenberg Matrices

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In recent years there has been a growing interest in companion matrices: matrices A of order n, that have (i) $n^2 - n$ entries that are constants; (ii) the n remaining entries of A are the variables $x_1, \dots, x_n$; and (iii) the characteristic polynomial of A is $\lambda^n - x_1\lambda^{n-1} - \cdots - x_{n-1}\lambda - x_n$.

Sparse companion matrices (with only $2n - 1$ nonzero entries) are well understood: every sparse companion matrix is equivalent to a Hessenberg matrix of a particular simple type. Recently, Deaett et al. [2] started the systematic study of non-sparse companion matrices (with more than $2n - 1$ nonzero entries). Our aim is to advance this study. They proved that every non-sparse companion matrix is nonderogatory, although not necessarily equivalent to a Hessenberg matrix. Nonetheless, companion matrices which are Hessenberg play an important role, to begin with, the Fiedler companion matrices are of this type. The variables in a Fiedler companion matrix form a "ladder" that starts in position $(n, 1)$ with the x_n variable and ends in a diagonal position (i_1, i_1) with the x_1 variable; these two positions define the so called i_1-block. In this talk, the non-sparse companion matrices which are unit Hessenberg are studied: they are companion, have ones in the superdiagonal, and zeros above the superdiagonal.

An intriguing open question was stated by Deaett et al. [2]: "*We wonder if, in producing a companion matrix by changing some zero entries of a Fiedler companion matrix $F_{i_1,\dots,i_n}$ by nonzero constants, the extra nonzero entries are always restricted to the submatrix corresponding to the i_1-block*". They partially confirmed that supposition.

Theorem 5.4 [2]: Let A be a matrix obtained from the Fiedler companion matrix $F_{i_1,\dots,i_n}$ by changing zero entries that are not in the i_1-block. Then A is not companion.

We make some progress in this problem by solving the case in which the change of zero entries in the Fiedler companion matrix is only made below the superdiagonal.

It remains unknown if exists a companion matrix which is obtained from some Fiedler companion matrix $F_{i_1,\dots,i_n}$ by changing at least one zero entry of the i_1-block and at least one zero entry above the superdiagonal.

We will discuss some other related results that appear in [1].

This is joint work with Alberto Borobia (Universidad Nacional de Educación a Distancia, UNED). Supported by the Agencia Estatal de Investigación *of Spain through grants PID2019-106362GB-I00/AEI/10.13039/5 and MTM2017-90682-REDT.*

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Companion pencils for scalar (and matrix) polynomials in the monomial basis

Fernando De Terán

Universidad Carlos III de Madrid

In this talk, we consider general companion pencils for scalar polynomials (given in the monomial basis) over arbitrary fields. More precisely, if

$$p(z) = \sum_{i=0}^{n} a_i z^i \tag{4}$$

is a scalar polynomial, with $a_i \in \mathbb{F}$, for $0 \leq i \leq n$, and $\mathbb{F}$ being an arbitrary field, then a *companion pencil* is of the form $L(z) = A + zB$, with A and B being $n \times n$ matrices with entries in the ring of polynomials in $a_0, \cdots, a_n$ (namely $\mathbb{F}[a_0, \cdots, a_n]$) satisfying

$$\det L(z) = \alpha p(z), \qquad \text{for some } \alpha \in \mathbb{F}. \tag{5}$$

We will first show several well-know classes of companion pencils, and then we will present some theoretical results about general companion pencils, like:

- The Smith form of every companion pencil is $\left[\begin{smallmatrix} I_{n-1} & 0 \\ 0 & \frac{1}{a_n}p(z) \end{smallmatrix}\right]$.
- Companion pencils are nonderogatory.

We will also pay attention to the sparsity. In particular, by imposing some natural restrictions on the entries, we determine the smallest possible number of nonzero entries in any companion pencil.

If time permits, we will also show how the notion of companion pencil is extended to matrix polynomials, and analyze some of the previous questions for this notion as well.

Most of this talk is based on [1] and [2].

This is joint work with Carla Hernando. Supported by Ministerio de Economía y Competitividad of Spain through grants MTM2017- 90682-REDT and MTM2015-65798-P, and by Agencia Estatal de Investigación of Spain through grant PID2019-106362GB-I00/AEI/10.13039/501100011033.

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MS-16: Riordan Arrays and Related Topics

Organisers: Gi-Sang Cheon (Sungkyunkwan University, South Korea), Tian-Xiao He (Illinois Wesleyan University) and Paul Barry (WIT, Ireland)

Theme: The Riordan group was first defined by Shapiro et al in 1991. This matrix group and its generalizations have many applications, covering such diverse areas as combinatorial identities, lattice path enumeration, and special functions and orthogonal polynomials. The study of this Frechet-Lie group, its subgroups and its elements of finite order are also areas of current research. Riordan arrays have also found applications in such areas as graph theory and partially ordered sets, where the notions of Riordan graphs and Riordan posets have been defined. Riordan arrays are lower triangular matrices with interesting structural properties in their own right. Many of these properties are related directly to the algebra of the power series that define the matrices.

Enumerative results for connected bipartite non-crossing geometric graphs

Minho Song

AORC, Sungkyunkwan University

In this talk, we present enumeration problems for geometric graphs which are connected bipartite non-crossing graphs (CBN graphs for short) with n+1 points in convex position. We introduce a production matrix for such geometric graphs, and a formula for the number of connected bipartite graphs, which gives an answer to an open question posed at [1]. We also construct a graph operation, which we call *odd-cycle removal*, to obtain a generating tree for CBN graphs. For the last, we show a recurrence relation for the number of CBN graphs by using the characteristic polynomial of the production matrix.

This is joint work with Gi-Sang Cheon, Hong Joon Choi, and Guillermo Esteban.

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On claw-free Toeplitz graphs

Bumtle Kang

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An $n \times n$ matrix $T = (t_{ij})_{1 \le i,j \le n}$ is called a *Toeplitz matrix* if $t_{i,j} = t_{i+1,j+1}$ for each $i, j = 1, \ldots, n-1$. Toeplitz matrices are precisely those matrices that are constant along all diagonals parallel to the main diagonal, and thus a Toeplitz matrix is determined by its first row and column.

A *Toeplitz graph* $G = (V, E)$ is a undirected graph with a symmetric Toeplitz adjacency matrix $A(G)$. i.e. identical on all its diagonals parallel to the main diagonal of $A(G)$. A Toeplitz graph G is therefore uniquely defined by the first row of $A(G)$, a $(0,1)$-sequence. If the 1's in the first row of a Toeplitz matrix are placed at positions $1 + t_1$, $1 + t_2$, $\ldots$, $1 + t_k$ with $1 \le t_1 < t_2 < \ldots < t_k < n = |V|$, we may simply write $T_n\langle t_1, t_2, \ldots, t_k\rangle$, two vertices x, y being connected by an edge iff $|x - y| \in \{t_1, t_2, \ldots, t_k\}$.

This study was initiated by the observation that chordal Toeplitz graphs $T_n\langle t_1, \ldots, t_k\rangle$ with $n > t_{k-1} + t_k$ are claw-free. A claw in a graph means a star $K_{1,3}$ as an induced subgraph. We find an interesting family of claw-free Toeplitz graphs so called 'cocoonery' and show that if $n > t_{k-1} + t_k$, then this family becomes exactly the family of claw-free Toeplitz graphs. We also completely characterize a claw-free Toeplitz graph $T_n\langle t_1, \ldots, t_k\rangle$ for $k = 2$ and $k = 3$. We go further to study Toeplitz graphs which happen to be line graphs.

A Riordan Array Approach to Some Problems involving Lattice Paths, Trees and Partitions

Naiomi T. Cameron

Spelman College

A Riordan array is an infinite lower triangular matrix that is determined by a pair (g, f) of generating functions meeting certain conditions [4]. With the right conditions for g and f, Riordan arrays can be used with great effect to study many types of combinatorial problems, including the enumeration of lattice paths, rooted plane trees and noncrossing partitions [1, 2, 3]. Moreover, since Riordan arrays form a group, there is an algebraic structure out of which new combinatorial insights can be drawn. This talk will relate a number of combinatorial problems about lattice paths, partitions, RNA secondary structures and plane trees to algebraic structure in the Riordan group.

This is joint work with Asamoah Nkwanta (Morgan State University).

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Competition periods and matrix periods of Boolean Toeplitz matrices

Homoon Ryu

Seoul National University, Applied Algebra and Optimization Research Center

Given subsets S and T of $\{1,\ldots,n-1\}$, an $n\times n$ Toeplitz matrix $A=T_n\langle S;T\rangle$ is defined to have 1 as the (i,j)-entry if and only if $j-i\in S$ or $i-j\in T$. In this talk, we present our results on matrix periods and competition periods of Toeplitz matrices over a binary Boolean ring $\mathbb{B}=\{0,1\}$. We showed that if $\max S+\min T\le n$ and $\min S+\max T\le n$, then A has the matrix period d/d' and the competition period 1 where $d=\gcd(s+t\mid s\in S, t\in T)$ and $d'=\gcd(d,\min S)$. Moreover, we could show that the limit of the matrix sequence $\{A^m(A^T)^m\}_{m=1}^{\infty}$ is a directed sum of matrices of all ones except zero diagonal. In many literatures we see that graph theoretic method can be used to prove strong structural properties about matrices. We also proceeded our work from a graph theoretic point of view.

This is joint work with Gi-Sang Cheon (Sungkyunkwan University, AORC), Bumtle Kang (AORC), and Suh-Ryung Kim (Seoul National University, AORC). This work was partially supported by Science Research Center Program through the National Research Foundation of Korea(NRF) Grant funded by the Korean Government (MSIP)(NRF-2016R1A5A1008055). G.-S. Cheon was partially supported by the NRF-2019R1A2C1007518. Bumtle Kang was partially supported by the NRF-2021R1C1C2014187. S.-R. Kim was partially supported by the Korea government (MSIP) (NRF-2017R1E1A1A03070489).

A Recursive Relation Approach to Riordan Arrays

Tian-Xiao He

Illinois Wesleyan University

A recursive relation approach to Riordan arrays is introduced. This approach gives a representation of the entries of a Riordan array (g, f) in terms of recursive linear combinations of the coefficients of g. On the other hand, Riordan arrays provide a unified way to construct the identities of linear recursive sequences of arbitrary orders with arbitrary initial conditions. Some related topics such as Gaussian binomial coefficients, interpolation, and q-analogs of Riordan arrays in terms of linear recursive sequences are also discussed.

Riordan posets and associated matrix algebras

Gukwon Kwon

Sungkyunkwan University

One may think of a new class of partially ordered sets represented as binary Riordan matrices referred to as 'Riordan posets'. This notion extends the theory of Riordan matrices into the domain of poset theory. In this talk, we establish the criterion for a given binary Riordan matrix to be defined as a Riordan poset matrix. It is also shown that every Riordan poset is a locally finite poset. This leads to the construction of various matrix algebras obtained from incidence algebras of Riordan posets. Many structural properties of Riordan posets are studied and various families of Riordan posets are introduced. A class of series-parallel posets is derived by extending the notion of Riordan posets to include exponential Riordan matrices, and it is obtained from Sheffer sequences of classical orthogonal polynomials.

This is joint work with Gi-Sang Cheon, Bryan Curtis and Arnauld Mesinga Mwafise.

Set coverings

Emanuele Munarini

Politecnico di Milano

In 1966, Comtet wrote a short paper [1, 2] where he showed that the number of certain finite mathematical structures are linked by a combinatorial relation. First, he proved that the number of coverings and the number of filter basis of a finite set N can both be expressed in terms of the number of families of non-empty subsets of N. Then, he showed that the number of topologies of N can be expressed in terms of the number of T_0-topologies of N.

In the first part of this talk, we will review the above results in the context of combinatorial species [4], emphasizing the fact that the mentioned relations hold also at the level of combinatorial objects and not only at a numerical level. Then, in the second part of the talk, we will focus on coverings and minimal coverings [3], and we will present some combinatorial and algebraic properties for the associated polynomials. Some of these results are obtained by using Sheffer matrices.

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Pseudo-involutions and palindromes in the Riordan group

Lou Shapiro

Howard University

Recently, several papers have come out linking pseudo-involutions and palindromes. This presentation is an introduction to some of the main ideas, examples, and uses of the connection of these two topics. The more direct result is that if A, B, C are pseudo-involutions so are $ABA, BAAB, ABCBA$, and any other palindromic combination. We present a few examples and then move onto the second connection.

A common functional relation for many combinatorial generating functions is $g = 1 + z\gamma(g)$. Then the pseudo-involutory companion for g is $f = z\frac{\gamma(g)}{g\gamma\left(\frac{1}{g}\right)}$. The simplest case occurs when $\gamma(g) = \gamma(1/g)\, g^d$, so that $\gamma(g)$ is a palindrome. In that case $f = zg^{d-1}$. Then for no extra work we know what f is, and we have a pseudo-involution $\left(g, zg^{d-1}\right)$ in the $(d-1)$-Bell subgroup. For instance, for the Catalan generating function $\gamma(z) = z^2$, so $\gamma(z)/\gamma(1/z) = z^4$ and $f = zC^{4-1}$ so that $\left(C, zC^3\right)$ is a pseudo-involution in the 3-Bell subgroup.

Going further leads to the twin-power theorem, B-functions, using Chebyshev polynomials to compute B-functions, and the pseudo-enhancement theorem.

Commutators in the Riordan group

Ana Luzón

Universidad politécnica de Madrid

In this talk, I will present some results related to commutators in the Riordan group. I will describe some particular examples involving involutions, pseudo-involutions or, generally, reversible elements. To do that, I will use a special nested sequence of normal subgroups which are really the Riordan version of some subgroups of the substitution group of formal power series introduced by Jennings in [1]. This work is based in our recent preprint [2].

This is joint work with M. A. Morón (UCM) and L.F. Prieto-Martínez (UPM). Supported by the Spanish goverment, Grant PGC2018-098321-B-I00.

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Quasi-involutions of the Riordan group

Nikolaos Pantelidis

South East Technological University

A Riordan quasi-involution is an aerated Riordan matrix whose inverse contains the exact same entries with $\pm$ signs on alternating non-zero subdiagonals [1, 2]. In this talk, we discuss about the quasi-involutions as combinatorial and algebraic objects in Riordan array research [4].

Expanding the concept of a quasi-involution to $k-$leveled aerated matrices, for $k > 1$, we analyse these elements that satisfy the quasi-involution property. From a combinatorial point of view, we present structural properties of these elements. We link them to known Riordan subgroups, and by introducing the theory of quasi-compressions, we prove a factorization theorem for a certain family of Riordan quasi-involution. Finally, we discuss the importance of quasi-involutions in the Heisenberg-Weyl algebra [3].

This is joint work with Aoife Hennessy (South East Technological University) and Paul Barry (South East Technological University).

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MS-17: Linear Algebra for Designs and Codes

Organisers: Ronan Egan (Dublin City University), Ilias Kotsireas (Wilfrid Laurier University), Pardaig Ó Catháin (Dublin City University) and Eric Swartz (College of William and Mary)

Theme: Linear algebra is an essential tool in the study of designs and codes: the Bruck-Ryser-Chowla theorem uses invariants of quadratic forms to give the best known non-existence results for designs. Delsarte's theory of association schemes and linear programming bounds is central in the study of coding theory. This mini-symposium will bring together researchers in design theory and coding theory to discuss old and new results in the field with an emphasis on linear algebraic techniques.

Cocyclic Two-Circulant Core Hadamard Matrices

Santiago Barrera Acevedo

Monash University

The two-circulant core (TCC) construction for Hadamard matrices (HMs) uses two sequences with almost perfect autocorrelation to construct a HM. A research problem of K. Horadam asks whether such matrices are cocyclic. Using ideas from permutation groups, we prove that the order of a cocyclic TCC HM coincides with the order of a HM of Paley type, Sylvester type or certain multiples of these orders. In addition, we show that there exist cocyclic TCC HMs at all allowable order less or equal to 1000 with at most one exception.

This is joint work with Padraig Ó Cathaín and Heiko Dietrich.

On some constructions of divisible design Cayley graphs and digraphs

Andrea Švob

University of Rijeka

Haemers, Kharaghani and Meulenberg have defined divisible design graphs (DDGs for short) as a generalization of (v, k, λ)-graphs (see [4]). Divisible design digraphs, a directed graph version of divisible design graphs, were introduced in [1]. Let G be a group and S a subset of G not containing the identity element of the group, which will be denoted by e. The vertices of the Cayley digraph $Cay(G, S)$ are the elements of the group G, and its arcs are all the couples (g, gs) with $g \in G$ and $s \in S$. In this talk we will present some constructions of divisible design Cayley graphs and digraphs that were studied in [2] and [3].

This is joint work with Dean Crnković (University of Rijeka) and Hadi Kharaghani (University of Lethbridge). Supported by the Croatian Science Foundation, Grant 6732.

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The Maximal Determinant Problem and Generalisations

Guillermo Nuñez Ponasso

Worcester Polytechnic Institute

A corollary to *Hadamard's inequality* states that every $n \times n$ matrix H with complex entries of modulus 1 satisfies the determinant inequality $|\det(H)| \leq n^{n/2}$. If H meets the bound with equality, then H is an *Hadamard matrix* and $HH^* = nI_n$. The case where the entries of H are ± 1 is well-studied. Here the Hadamard bound is only achievable when $n = 1, 2$ or a multiple of 4. In general one may ask what the maximum absolute value of the determinant of a ± 1 matrix of order n is. Matrices achieving this maximum are known as *maximal determinant matrices* of *D-optimal designs*, which are applied in the statistical theory of experimental designs. There are sharpened upper bounds for the determinant when $n \geq 3$ is not divisible by 4, which depend on the *residue class* of n modulo 4.

In this talk, we consider a generalisation of the maximal determinant problem to the case of matrices with entries taken from the set of k-th roots of unity. As in the real case, an Hadamard matrix with entries in the k-th roots of unity, known as a *Butson-Hadamard matrix*, saturates Hadamard's bound. Such matrices do not always exist, however. We will present new upper and lower bounds for the maximal value of the determinant in the case of third, fourth and sixth roots of unity. These are precisely the cases when the k-th roots generate a lattice in $\mathbb{C}$, which allows us to generalise previously-known upper bounds from the real case. Finaly we present results for lower bounds obtained from matrices in the *Bose-Mesner algebra* of *strongly regular graphs* and *cyclotomic association schemes.*

Perfect 1-factorisations and Hamiltonian Latin squares

Ian Wanless

Monash University

A 1-*factorisation* of a graph is a decomposition of the edges of the graph into 1-factors (perfect matchings). The 1-factorisation is *perfect* if the union of any two of its 1-factors is a Hamilton cycle. A P1F of the complete bipartite graph $K_{n,n}$ is equivalent to a *row-Hamiltonian Latin square* of order n. These are Latin squares with no non-trivial Latin subrectangles; equivalently, the permutation which maps any row to any other row is an n-cycle. Each Latin square has six *conjugates* obtained by uniformly permuting its (row, column, symbol) triples. Let $\nu(L)$ denote the number of conjugates of L that are row-Hamiltonian. It is easy to see that $\nu(L) \in \{0, 2, 4, 6\}$ and that $\nu = 0$ can be achieved for all $n > 3$. At the other extreme, $\nu = 6$ is achieved by the so-called *atomic* Latin squares, including the Cayley tables of cyclic groups of prime order. There is also a known infinite family with $\nu = 2$. By converting our problem into linear algebra, we were able to find the first infinite family with $\nu = 4$. We can build Latin squares in which *every* pair of rows form a Hamilton cycle and *no* pair of columns form a Hamilton cycle. As a corollary, we answer a question on quasigroup varieties posed by Falconer in 1970.

This is joint work with Jack Allsop (Monash)

The Density of Complementary Subspaces

Ferdinand Ihringer

Universiteit Gent

Let V be a finite vector space of dimension $d = e + e'$ over the field with q elements. Consider a family Y_1 of e-spaces and a family Y' of e'-spaces with positive density of at least $C^{-1}q^{1-\frac{d}{2}}$ each. We show, using an easy argument relying on the expander mixing lemma and well-known properties of the irreducible modules of Grassmann graphs, that the probablity of $S_1 \cap S_2 = \{0\}$ for $(S_1, S_2) \in Y_1 \times Y_2$ is at least $1 - (C+1)q^{-1} + (C-1)q^{-2}$.

Our motivation is as follows: Suppose that V is equipped with a nondegenerate reflexive sesquilinear form σ. Let Y_1 and Y_2 be the families of nondegenerate subspaces with respect to σ. Using long and sophisticated geometric arguments it is shown in [1] that the probability of $S_1 \cap S_2 = \{0\}$ is at least $1 - Cq^{-1}$ for relatively small C, while leaving a few cases open. Our linear algebra technique takes care of the open cases in [1], tends to improve C, and avoids any deep dives into geometric arguments.

This is joint work with Stephen Glasby (University of Western Australia) and Sam Mattheus (Vrije Universiteit Brussel).

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q-Polymatroids and Designs over $GF(q)$

Eimear Byrne

University College Dublin

A q-polymatroid consists of a lattice of subspaces of a vector space endowed with a rank function that is both increasing and submodular. They were studied independently by Gorla *et al* (2020) and Shiromoto (2019) as q-analogues of polymatroids and in reference to matrix codes. A number of invariants of codes are in fact matroid invariants, including the MacWilliams duality theorem. MacWilliams identities for classical matroids have been studied by a number of authors (e.g. Brylawski, Oxley, Britz, Shiromoto). In this talk we will consider duality of q-polymatroids and will give a version of a MacWilliams theorem for q-polymatroids, using the characteristic polynomial. As as application of this result, we will state an Assmus-Mattson-like theorem that establishes criteria for the existence of weighted subspace designs arising from a q-polymatroid.

This talk is based on joint work with M. Ceria, R. Jurrius, and S. Ionica.

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Enumeration of some matrices and free linear codes over finite commutative rings

Siripong Sirisuk

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Let R be a finite commutative ring with identity. A row of single unit in R^n is a row in which a single entry is a unit and all other entries are zero. Two enumeration problems over R are presented. We enumerate the matrices over R with a given McCoy rank and a given number of rows of single unit. We also enumerate the free linear codes over R which have a given rank and a given number of standard basis vectors.

This study was supported by Thammasat University Research Fund, Contract No. TUFT 27/2565.

q-ARY STRONGLY REGULAR GRAPHS

Dean Crnković

University of Rijeka

The notion of q-analog of designs has been introduced by Delsarte [3]. In 1987, Thomas [4] constructed the first non-trivial q-analog of design with parameters 2-$(n, 3, 7; 2)$, $n > 6, n = 6k + 1$ or $n = 6k - 1$. An important result was given in [2], where the authors constructed a design over a finite field with parameters 2-$(13, 3, 1; 2)$ which was the first known example of a Steiner q-design that does not arise from spreads. In this talk we will introduce the notion of q-analog of strongly regular graphs, given in [1], and present some new results.

This is joint work with Michael Braun (Darmstadt University of Applied Sciences), Maarten De Boeck (University of Rijeka), Vedrana Mikulić Crnković (University of Rijeka) and Andrea Švob (University of Rijeka). Supported by the Croatian Science Foundation, Grant 6732.

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Negacyclic weighing matrices

Robert Craigen

University of Manitoba

A matrix obtained by negating every entry of a circulant matrix below the diagonal is said to be negacyclic. Negacyclic structure, as with its cousin circulant structure, arises remarkably often in questions connected to Hadamard, weighing matrices and generalizations thereof, but in comparison to that for circulants, the literature devoted specifically to this structure has been sparse and fragmented. There are numerous comprehensive surveys on the state of knowledge of circulant weighing matrices, including at least three separate graduate theses having this title: *Circulant weighing matrices*, and at least two others including that exact phrase, besides a sizeable literature devoted entirely to the special case of circulant Hadamard matrices ... but there is no analog for negacyclic case.

We develop some basic theory of negacyclic weighing matrices both in aspects parallel to circulant matrices and those in which the two types diverge. We will also discuss the results of an empirical examination of the existence of negacyclic weighing matrices of small orders (up to 52) and square weights up to 100.

This talk is includes work carried out with Undergraduate Research Students Ted Eaton, Colin Desmarais, Peter Naylor, Ian Thompson, William Kellough and Dana Kapoostinsky during 2013–2020 within USRA programs funded by NSERC and the University of Manitoba, and partially supported by an NSERC grant.

MS-18: Kemeny's constant on networks and its application

Organisers: Ángeles Carmona, Maria José Jiménez and Margarida Mitjana

Theme: The computation of the **Kemeny's constant** is a classical problem in the theory of **Markov chains** and has multiple applications. The different ways to afford the problem go from Linear Algebra to discrete Potential Theory. The mean first passage time is closely related to other well–known metrics for graphs and Markov chains. First, the **Kirchhoff index**, also known as the **effective graph resistance**, is a related metric quantifying the distance between pairs of vertices in an electric network. The relationship between **electrical networks** and **random walks** on graphs is well–known. For an arbitrary graph, the Kirchhoff index and the Kemeny constant can be calculated from the eigenvalues of the conductance matrix and the transition matrix, respectively. This minisymposium will give an opportunity to communicate the latest developments in the area and its applications presenting some current research and stimulating new ideas and collaborations, as well as bringing some highlights to its classical properties.

G-inverses for random walks

Àlvar Martín

Universitat Politècnica de Catalunya - BarcelonaTech

In terms of random walks skills, if we asume that the system is in an initial state s_i, the number of expected steps to reach state s_j is described by the so-called Mean First Passage Time (MFPT), which is denoted by m_{ij}. The matrix characterizing the MFPT can be written in terms of the g-inverses of the combinatorial Laplacian, see [1].

Although the MFPT is an element that allows to describe random walks, it is not the only one. It is well known that the time to reach a random state s_j, starting from an initial state s_i, is a constant that does not depend of the initial state. This time is the so-called Kemeny's constant, that it can be expressed in terms of g-inverses of the above-mentioned Laplacian.

In this communication we will obtain expressions both for the MFPT and for the Kemeny's constant in terms of g-inverses of the combinatorial Laplacian. In addition, as an application, we introduce the case of the star.

This is joint work with Ángeles Carmona and Maria José Jiménez (UPC). Partially supported by the Departament de Matemàtiques of UPC.

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An edge centrality measure based on the Kemeny constant

Federico Poloni

University of Pisa

We introduce a centrality measure $c(e)$ for the edges e of an undirected graph G. It is based on the variation of the Kemeny constant of the graph after removing the edge e, following an idea introduced in [1]. The new measure is designed to ensure non-negativity, avoiding the so-called Braess paradox [2]. We introduce an optimized numerical method to compute it, and a regularization technique to deal with cut-edges and disconnected graphs. Numerical experiments performed on synthetic tests and on real road networks show that this measure is particularly effective in revealing bottleneck roads whose removal would greatly reduce the connectivity of the network.

This is joint work with D. Altafini, D. A. Bini, V. Cutini, and B. Meini (University of Pisa, Department of Mathematics and Department of Energy Engineering, Systems, Land and Buildings). Supported by the University of Pisa, grant PRA-2022-61 and by INDAM/GNCS.

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Mean first passage time for distance-biregular graphs

María José Jiménez

Universitat Politècnica de Catalunya-BarcelonaTech

In this presentation, we obtain the explicit expression for the Group inverse of the Laplacian matrix associated with distance–biregular graphs, ([1]). A bipartite graph is called distance–biregular (DBR) if all the vertices of the same partite set admit the same intersection array. So, this kind of graphs are characterized by having two intersection arrays instead of one as in the case of distance-regular graphs. Examples of this kind of graphs are complete bipartite graphs, subdivision graphs of minimal cages and some block designs, see [3]. As an application, we provide the mean first passage time for DBR graphs as well as the Kemeny constant. The above expression will be given in terms of the so–called equilibrium measure for a vertex $\{x\}$, see [2]. Finally, we provide some examples as star graphs.

This is joint work with Á. Carmona and A.M. Encinas (UPC). Partially supported by the Departament de Matemàtiques of UPC.

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Schrödinger random walks and mean first passage time generalization

Ángeles Carmona

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For spreading and diffusion processes, Random Walks (RW) represents a mathematical model and can be used to extract information about important issues in networks. In the study of networks, one often seeks to rank nodes, edges, or other structures based on their relative importance (centrality measures). RW are characterized from the transition probability matrix, that is the probabilistic counterpart of the combinatorial Laplacian or even of the normalized Laplacian. The short–term behavior of a RW can be studied trough the so–called mean first passage time from one state to another. The mean first passage time from a given state is the solution of a Poisson problem with respect to the probabilistic Laplacian and hence, from a matrix point of view, it can be obtained from some generalized inverses of the probabilistic Laplacian. On the other hand, it is known that the expected time to get any randomly chosen vertex from a given one is constant and independent of the starting vertex. The common value is called Kemeny's constant. The computation of Kemeny's constant is a classical problem in the theory of RW and the different ways to afford the problem go from Linear Algebra to discrete Potential Theory. As a consequence of our works in the context of BVP on networks, we have obtained some results that express the mean first passage time in terms of equilibrium measures for the combinatorial Laplacian, [1].

All the above models are based on the hypothesis that in each step the random walk move from one node to another different one. Only the so-called Lazy Random Walks contemplate the probability of remaining at a state, but this probability is always constant and usually equal to $1/2$. Therefore, they are far away to include all the real situations that would be modeled in this context. Assigning a different positive transition probability to each node will include the probability to remain in each state, depending on the state, and suppose a challenge in RW Theory. Our previous work has proved that this assumption is consistent with the consideration of general Schrödinger operators (M-matrices), [2]. The transition probability associated to a state will be defined through the potential at the associated node. So, we can consider a new type of RW that could be called Schrödinger random walk (SRW). We define the fundamental parameters, such as the mean first passage time, in the case of SRW, which will provide new properties and information. Moreover, they can be obtained from the solution of boundary value problems for the Schrödinger operators.

This is joint work with A.M. Encinas, M.J. Jiménez, M. Mitjana and À. Martín (UPC). Partially supported by the Departament de Matemàtiques of UPC.

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The resistance magnitude of a graph

Karel Devriendt

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In [1], Tom Leinster introduced the *magnitude* as an invariant of enriched categories – this is a general class of objects that includes finite metric spaces. In many cases, magnitude behaves similar to the Euler characteristic or cardinality, and in the case of metric spaces it can be thought of as the 'effective number of points' at a given scale.

Let (X, d) be a finite metric space with the *similarity matrix* at scale $t > 0$ given by $Z_t := (\exp(-d(i,j)t))$. If this matrix is invertible, the magnitude is defined as

$$|Z_t| := \mathbf{1}^T Z_t^{-1} \mathbf{1}.$$

Leinster studied the magnitude for the vertices of a graph with the shortest-path distance as a metric (V, d) in [**?**]. If instead we consider the vertices of a graph with the *effective resistance* as a metric (V, ω), then the leading term of magnitude equals

$$\lim_{t\to 0} \frac{|Z_t| - 1}{t} = 2\sigma^2.$$

We call σ^2 the *resistance magnitude* of a graph and with resistance matrix $\Omega = (\omega(i,j))$, this has the following equivalent definitions

$$2\sigma^2 = \left(\mathbf{1}^T \Omega^{-1} \mathbf{1}\right)^{-1} = \max_{\mathbf{f}^T \mathbf{1} = 1} \mathbf{f}^T \Omega \mathbf{f}.$$

The resistance magnitude appears to be a rich graph invariant with many properties: it is related to discrete curvature (in the sense introduced in [3]), it has certain inclusion-exclusion and submodularity properties and it is the squared radius of the Euclidean embedding of $(V, \sqrt{\omega})$, see [4].

The resistance magnitude relates to other well-known resistance-based graph invariants such as the Kirchhoff index (R_G) and Kemeny's constant (K), as

$$R_G \leq \sigma^2/n^2 \text{ and } K \leq \sigma^2/m$$

where $n = |V|$ is the number of vertices and $m = |E|$ the number of edges (or total edge weight for weighted graphs); equality is achieved in both cases for vertex-transitive graphs.

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Biased Advection operators on undirected graphs

Manuel Miranda

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In certain real-world scenarios it is important to account for the influence of nearest neighbors on the diffusion of a particle located at a given node of an undirected graph. To capture this influence the so-called hubs-biased graph Laplacians were proposed in [2]. We investigated the self-adjoint of these operators and discovered that they correspond to operators describing advective processes, where a degree-based drift pulls/push the diffusive particle from/towards the hubs of the network. Advection operators were previously defined only for digraphs, where the direction of the edges ruled the drift, but the new operators that we present here act on undirected graphs. The process controlled by this operators converges towards an ordered state in which the final concentration of the nodes depends on the degree of each node.

In this talk, we will explain how this new advective operators in undirected graphs are constructed, which properties do they have and which is its final configuration. Moreover, we will construct an advection-diffusion equation in which both processes "compete" in a graph. We will show the analytic expression of the steady state of this kind of processes. Finally, we will illustrate the current ideas studying how advection-diffusion shapes movement of the species *L. catta* when the foraging occurs in a very patched landscape network in Southern Madagascar.

This is joint work with Ernesto Estrada (IFISC). Supported by the scholarship PRE2020-092875 by MCIN/AEI/10.13039/501100011033 and by FSE invierte en tu futuro.

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Directed forests and the constancy of Kemeny's constant

Steve Kirkland

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Consider a discrete-time, time-homogeneous Markov chain on states $1, \ldots, n$ whose transition matrix is irreducible. Denoting the mean first passage times by $m_{jk}, j, k = 1, \ldots, n$ and the stationary distribution vector entries by $v_k, k = 1, \ldots, n$ a remarkable result of Kemeny reveals that the quantity $\sum_{k=1}^{n} m_{jk} v_k$ does not depend on the choice of j. In this talk, we consider $\sum_{k=1}^{n} m_{jk} v_k$ from the perspective of algebraic combinatorics, and provide an intuitive explanation for its independence on the choice of the state j. The all minors matrix tree theorem is the key tool employed.

Kemeny's constant for non-backtracking random walks

Jane Breen

Ontario Tech University

Kemeny's constant for a connected graph G is the expected time for a random walk to reach a randomly-chosen vertex u, and is a quantity independent of the choice of the initial vertex. We extend the definition of Kemeny's constant to non-backtracking random walks and compare it to Kemeny's constant for simple random walks. We explore the relationship between these two parameters for several families of graphs and provide closed-form expressions for regular and biregular graphs. In nearly all cases, the non-backtracking variant yields the smaller Kemeny's constant.

This is joint work with Nolan Faught, Cory Glover, Mark Kempton, Adam Knudson, and Alice Oveson (Brigham Young University). Supported by NSERC Discovery Grant RGPIN-2021-03775.

Kemeny's Constant for Several Families of Graphs and Real-world Networks

Robert E. Kooij

Delft University of Technology, the Netherlands

The linear relation between Kemeny's constant, a graph metric directly linked with random walks, and the effective graph resistance in a regular graph has been an incentive to calculate Kemeny's constant for various networks. In this paper we consider complete bipartite graphs, (generalized) windmill graphs and tree networks with large diameter and give exact expressions of Kemeny's constant. For non-regular graphs we propose two approximations for Kemeny's constant by adding to the effective graph resistance term a linear term related to the degree heterogeneity in the graph. These approximations are exact for complete bipartite graphs, but show some discrepancies for generalized windmill and tree graphs. However, we show that a recently obtained upper-bound for Kemeny's constant in [1] based on the pseudo inverse Laplacian gives the exact value of Kemeny's constant for generalized windmill graphs. Finally, we have evaluated Kemeny's constant, its two approximations and its upper bound, for 243 real-world networks. This evaluation reveals that the upper bound is tight, with average relative error of only 0.73%. In most cases the upper bound clearly outperforms the other two approximations.

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MS-20: Special Matrices

Organisers: Natália Bebiano (Universidade de Coimbra), Susana Furtado (Universidade do Porto) and Mikail Tyaglov (Shanghai Jiao Tong University)

Theme: The goal of this minisymposium is to spread recent developments, and stimulate new research, on structured matrices with applications in various fields of pure and applied science. Future collaborations among researchers will also be promoted. Particular attention will be given to tridiagonal, k-Toeplitz, Hankel, reciprocal, stochastic, Hurwitz and birth and death matrices. The use of these matrices in such areas as statistics, numerical analysis, engineering, economics and physics will be discussed.

Efficient vectors for perturbed consistent matrices

Susana Furtado

CEAFEL and Faculdade de Economia do Porto

An $n \times n$ matrix $A = [a_{ij}]$ is said to be a pairwise comparison matrix (PC matrix) or a reciprocal matrix if it is positive and $a_{ij} = \frac{1}{a_{ji}}$, for $i, j = 1, \ldots, n$. If, in addition, $a_{ij}a_{jk} = a_{ik}$, for $i, j, k = 1, \ldots, n$, then A is said to be consistent or transitive.

PC matrices and, in particular, consistent matrices, play an important role in the Analytic Hierarchy Process, a method used in Decision Making. In this method it may be important to approximate a PC matrix by a consistent one. In this context, the notion of efficient vector for a PC matrix arises.

A positive vector $w = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^T$ is said to be efficient for an $n \times n$ PC matrix $A = [a_{ij}]$ if there is no other vector $v = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T$ such that

$$\left| a_{ij} - \frac{v_i}{v_j} \right| \leq \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \text{for all } 1 \leq i, j \leq n,$$

with the inequality strict for at least one pair (i, j).

In this talk we describe all efficient vectors for a simple perturbed consistent matrix, that is, a PC matrix obtained from a consistent one by perturbing one entry above the main diagonal, and the corresponding reciprocal entry. As a consequence, we give a simple proof of the result obtained by K. Ábele-Nagy and S. Bozóki (2016) that states that any (positive) eigenvector of a simple perturbed consistent matrix associated with the Perron eigenvalue is efficient. Some related results for double and triple perturbed consistent matrices are also presented.

Based on a joint work with Henrique Cruz and Rosário Fernandes.

An algorithmic approach to the Symmetric Nonnegative Inverse Eigenvalue Problem

Richard Ellard

TU Dublin

Let $\sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a list of n real numbers. If there exists an $n \times n$ symmetric matrix A with nonnegative entries and spectrum σ, then we say σ is *symmetrically realisable*. The *Symmetric Nonnegative Inverse Eigenvalue Problem* (SNIEP) is the problem of characterising all symmetrically realisable lists.

Ellard and Šmigoc [1] showed that essentially all previously known sufficient conditions for symmetric realisability were equivalent; however, determining whether a given list of real numbers satisfies any of these equivalent conditions remained nontrivial. In this talk, I present an explicit algorithm to make this determination for a given list.

This is joint work with Helena Šmigoc (University College Dublin). Supported by Science Foundation Ireland, Grant 11/RFP.1/MTH/3157.

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A Low-complexity Algorithm to Uncouple the Mutual Coupling Effect in Antenna Arrays

Sirani M. Perera

Embry-Riddle Aeronautical University, USA

The interaction of electrical and magnetic fields between antenna array elements causes mutual coupling in multi-beam arrays. The presence of mutual coupling between array elements leads to variation in impedance, alteration in radiation patterns, changes in array characteristics, and noise coupling.

In this talk, we will utilize the structure of the mutual coupling matrix to obtain a sparse factorization followed by a low-complexity algorithm to reduce the mutual coupling effects. Next, signal flow graphs will be presented to show the connection of the algebraic operations associated with the proposed algorithm and to realize the system as an integrated circuit. Finally, the proposed fast algorithm will be exploited to digitally uncouple the mutual coupling effect in multi-beam antenna arrays.

This is joint work with Arjuna Madanayake (Florida International University, USA).

The periodic pseudo-Jacobi inverse eigenvalue problem

Natália Bebiano

University of Coimbra

The problem of reconstructing a *periodic pseudo-Jacobi matrix*, which is derived from the discretization and truncation of Schrödinger equation, arises in non-Hermitian quantum mechanics. Also the reconstruction of the Hamiltonian system of an indefinite *Toda lattice* and the symmetry reduction of the *Wess-Zumino-Novikov-Witten* model in quantum field theory are problems deserving the attention of physicists and mathematicians. In mathematics, this problem is referred to as *periodic pseudo-Jacobi inverse eigenvalue problem* (hereafter **PPJIEP**), and concerns the reconstruction from assigned spectral data of a specified periodic pseudo-Jacobi matrix Inspired in a discrete version of Floquet theory in a space with indefinite metric [Math. Comp. 35 (1980) 1203-1220] and a van Moerbeke's idea [Invent. Math. 37 (1976) 45-81], the **PPJIEP** problem is solved. We use two methods to characterize the signature operator so that the solution exists.

This is joint work with Wei-Ru Xu, Yi Gong and Guo -Liang Chen (China).

Sharp bounds on the least eigenvalue of a graph determined from edge clique partitions

Domingos M. Cardoso

Center for Research and Development in Mathematics and Applications - CIDMA, University of Aveiro

Sharp bounds on the least eigenvalue of an arbitrary graph are presented. Necessary and sufficient (just sufficient) conditions for the lower (upper) bound to be attained are deduced using edge clique partitions. As an application, we prove that the least eigenvalue of the n-Queens' graph $\mathcal{Q}(n)$ is equal to -4 for every $n \geq 4$ and it is also proven that the multiplicity of this eigenvalue is $(n-3)^2$. Additionally, some results on the edge clique partition graph parameters are obtained.

This is joint work with Inês Serôdio Costa (University of Aveiro) and Rui Duarte (University of Aveiro). Supported by the Center for Research and Development in Mathematics and Applications (CIDMA) which is financed by national funds through Fundação para a Ciência e a Tecnologia (FCT), Grant UIDB/04106/2020.

Self-adjoint operators associated with Hankel moment matrices

Christian Berg

University of Copenhagen, Denmark

Let $\mathcal{M}^*$ denote the set of positive measures with moments of any order and infinite support on the real line $\mathbb{R}$. The moment sequence of $\mu \in \mathcal{M}^*$ is denoted

$$m_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n = 0, 1, \ldots, \tag{6}$$

and we let

$$\mathcal{H} = \mathcal{H}_\mu = (m_{k+l})_{k,l=0}^{\infty} = \{m_{k+l}\} \tag{7}$$

denote the corresponding Hankel matrix.

Denoting by $\mathcal{F}$ the set of sequences $g \in \ell^2$ with only finitely many non-zero entries, we have a positive sesquilinear form Q defined on $\mathcal{F} \times \mathcal{F}$

$$Q(g, h) = \sum_{k,l=0}^{\infty} m_{k+l} g_k \overline{h_l}, \tag{8}$$

called the Hankel form associated with the sequence (m_n).

Widom proved in 1966 that the form $(Q, \mathcal{F})$ is bounded on ℓ^2 if and only if $m_n = O(1/n)$. In 2016 Yafaev proved that the form is closable if $m_n = o(1)$ and characterized the closure of the form based on his earlier work on quasi-Carleman operators. We give a new proof of the description of the closure based entirely on moment considerations. In 2020 Berg and Szwarc pointed out that the form $(Q, \mathcal{F})$ is also closable if μ is indeterminate of if μ is determinate with finite index of determinacy. We give a description of the self-adjoint operators $H = H_\mu$ (bounded or unbounded) in the Hilbert space ℓ^2 associated with the closed Hankel forms in the three cases mentioned, where the form is closable.

This is joint work with Ryszard Szwarc (Wrocław), see [1].

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Inequalities for means of matrices

Rute Lemos

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The axiomatic theory of operator connections and means was developped by F. Kubo and T. Ando [1]. Inequalities involving eigenvalues and singular values of Kubo-Ando means of matrices are surveyed and some log-majorization type results are deduced. Some inequalities for the singular values of Heinz means, which are not Kubo-Ando type means, are also obtained.

This is joint work with Graça Soares (CMAT–UTAD). Supported by Portuguese funds through the Center for Research and Development in Mathematics and Applications (CIDMA) and the Portuguese Foundation for Science and Technology (FCT), project UIDB/04106/2020.

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Some special matrices arising in computer vision and related optimization problems

João R. Cardoso

Polytechnic Institute of Coimbra – ISEC, and University of Coimbra – CMUC

This talk addresses two types of matrices that play an important role in Computer Vision: generalized essential matrices and sub-Stiefel matrices. We revisit their definition, the most relevant properties and discuss two related optimization problems of Procrustes-type whose solution involve those matrices. Effective algorithms for solving such problems are proposed and illustrative examples are provided.

This includes joint work with Pedro Miraldo (University of Lisbon, Portugal) and Krystyna Ziętak (University of Wroclaw, Poland). The speaker acknowledges the funding from Center for Mathematics, University of Coimbra, Portugal.

Tridiagonal matrices with two-periodic main diagonal

Mikhail Tyaglov

Shanghai Jiao Tong University

We find the spectrum of an arbitrary irreducible complex tridiagonal matrix with two-periodic main diagonal provided that the spectrum of the matrix with the same sub- and superdiagonals and zero main diagonal is known. Our result substantially generalises some of the recent results on the Sylvester-Kac matrix and its certain main principal submatrix.

This is joint work with Alexander Dyachenko (KIAM).

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MS-21: Tensors for signals and systems

Organisers: Kim Batselier (TU Delft), Philippe Dreesen (KU Leuven) and Bori Hunyadi (TU Delft)

Theme: The talks in this minisymposium will revolve around the application of tensor-based methods on various problems in signal processing, machine learning, and systems and control theory. The focus of the talks will be specifically on the different applications with the common thread being the explicit use of different tensor decompositions. Furthermore, we will host a number of talks on recent theoretical results on existence and uniqueness of tensor approximations.

Structured Tensor Decompositions in Functional Neuroimaging: Estimating the Hemodynamic Response

Borbala Hunyadi

Delft University of Technology

Functional neuroimaging techniques, such as functional magnetic resonance imaging (fMRI) or functional ultrasound (fUS) measure brain activity in a noisy and indirect way. Noisy, because they record a mixture of ongoing brain activity, physiological and non-physiological noise sources. Indirect, because they pick up the hemodynamic response: the changes in oxygen contentation (fMRI), volume and flow (fUS) of cerebral blood in response to neuronal activity. More specifically, this response (i.e. the measurement vector) is usually modelled as a convolution of the underlying activity (source vector) and the so-called hemodynamic response function (HRF). Source separation techniques that can extract the activity of interest along with the HRF are crucial for correctly interpreting the recorded data. In this talk, I will illustrate via two applications how tensor decompositions can solve this source separation problem. In the first application, EEG data simultaneously recorded with fMRI is also available, which provides information on the unknown source vector. The joint EEG-fMRI decomposition is formulated as a structured matrix-tensor factorization problem [1]. In the second application only fUS data is available. To tackle this ill-posed problem, we assume that the sources are uncorrelated. The resulting model - a convolutive mixture of uncorrelated sources - is formulated as a structured block term decomposition problem [2]. Both formulations lead to nonconvex optimization problems. I will discuss strategies to obtain robust results, using relevant constraints, model- and component-selection procedures. Finally, I will show that the structured tensor decompositions estimate the location (first application) and timing (second application) of the source of interest as well as the subject- and region-specific HRF in a biologically meaningful way.

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Tensor Decomposition of ECG Records for Persistent Atrial Fibrillation Analysis

Vicente Zarzoso

Université Côte d'Azur, CNRS, I3S Laboratory, Sophia Antipolis, France

Considered as the last great frontier of cardiac electrophysiology, atrial fibrillation (AF) is the most common sustained arrhythmia encountered in clinical practice, responsible for high hospitalization rates and a significant proportion of brain strokes in the Western world. Analyzing AF electrophysiological complexity noninvasively requires the extraction of the atrial activity (AA) signal from the electrocardiogram (ECG). To perform this task, most approaches including classical average beat subtraction need sufficiently long ECG records, thus limiting real-time analysis. Linear algebra techniques based on matrix factorizations can also be used for AA signal estimation by exploiting the spatial diversity of the multi-lead ECG, but require some constraints to guarantee uniqueness that may lack physiological grounds and hinder results interpretation.

This talk will review our recent results on multilinear algebra techniques such as tensor decompositions for noninvasive AA signal extraction in AF ECGs, which guarantee uniqueness under milder constraints on their factors. Specifically, the block term decomposition (BTD) has been shown to be particularly suitable to address this biomedical problem, as atrial and ventricular cardiac activity sources can be modeled by matrices with special structure. The structure of these matrices ensures model uniqueness while their rank is linked to signal complexity. In this framework, we have put forward the Hankel and Löwner BTD as AA extraction tools in AF ECG episodes, with validation in a population of persistent AF patients and several challenging types of ECG segments, including short beat-to-beat intervals and low-amplitude fibrillatory waves. Accurate AA extraction can be achieved from ECG segments as short as a single heartbeat. We have also developed a robust computational algorithm — the so-called alternating group lasso BTD (BTD-AGL) — to simultaneously recover the model structure (number of block terms and multilinear rank of each term) and the model factors. In addition, tensor modeling allows us to derive a novel index to quantify AF complexity nonivasively, useful to characterize stepwise catheter ablation, a first-line therapeutic option for the treatment of persistent forms of the arrhythmia. The index correlates with the expected decrease in AF complexity over ablation steps and is predictive of AF recurrence, which presents clear clinical interest.

Joint work with Pedro Marinho R. de Oliveira (BioSerenity, Paris, France) and Lucas Abdalah (Universidade Federal do Ceará, Fortaleza, Brazil). Work supported by the French government, through the 3IA Côte d'Azur Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR-19-P3IA-0002. V. Zarzoso holds the Chair "IAblation" from 3IA Côte d'Azur.

Multi-Tensor Decompositions for Personalized Cancer Medicine

Orly Alter

Scientific Computing and Imaging Institute and the Huntsman Cancer Institute at the University of Utah

Starting with our invention of the "eigengene," I will describe the formulation of physics-inspired multi-tensor generalizations of the singular value decomposition to (*i*) compare [1, 2, 3] and integrate any data types, of any number and dimensions, and (*ii*) scale with data sizes. These models (*iii*) are interpretable in terms of known biology and batch effects and (*iv*) correctly predict [4, 5] previously unknown mechanisms. By validating a genome-wide pattern of DNA copy-number alterations in brain [6] tumors as the best predictor of survival, our retrospective clinical trial [7] proved that the models (*v*) discover accurate, precise, and actionable genotype-phenotype relationships, (*vi*) are relevant to populations based upon whole genomes of small cohorts, and (*vii*) can be validated. We discovered this, and patterns in lung [8], nerve, ovarian, and uterine tumors, in public data. Such alterations were recognized in cancer, yet attempts to associate them with outcome failed, demonstrating that our algorithms are uniquely suited to personalized medicine.

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A quadratically convergent proximal algorithm for nonnegative tensor decomposition

Nico Vervliet

KU Leuven

The decomposition of tensors into simple rank-1 terms is key in a variety of applications in signal processing, data analysis and machine learning. While this canonical polyadic decomposition (CPD) is unique under mild conditions, including prior knowledge such as nonnegativity of the underlying factors can facilitate interpretation of the components. Inspired by the effectiveness and efficiency of Gauss–Newton (GN) for unconstrained CPD, we derive a proximal, semismooth GN type algorithm for nonnegative tensor factorization. Global convergence to local minima is achieved via backtracking on the forward-backward envelope function. If the algorithm converges to a global optimum, we show that Q-quadratic rates are obtained in the exact case. Such fast rates are verified experimentally, and we illustrate that using the GN step significantly reduces number of (expensive) gradient computations compared to proximal gradient descent.

This is joint work with Andreas Themelis (Kyushu University, Japan), Panagiotis Patrinos (KU Leuven, Belgium) and Lieven De Lathauwer (KU Leuven, Belgium). This work was supported by the Research Foundation Flanders (FWO) via projects G086518N, G086318N, and via postdoc grant 12ZM220N; KU Leuven Internal Funds via projects C14/18/068, C16/15/059, and IDN/19/014; Fonds de la Recherche Scientifique—FNRS and the Fonds Wetenschappelijk Onderzoek— Vlaanderen under EOS project No. 30468160 (SeLMA). This research received funding from the Flemish Government under the "Onderzoeksprogramma Artificiële Intelligentie (AI) Vlaanderen" program.

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Multi-task fMRI data fusion using Independent Vector Analysis and the PARAFAC2 tensor decomposition

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Paderborn University, Germany

The interest in data fusion, i.e., the joint analysis of multiple related datasets, has grown in recent years in various research areas, in particular, in biomedicine. Data-driven methods, especially methods based on joint matrix/tensor factorizations, have shown to be effective for data fusion [1]. Two of them are Independent Vector Analysis (IVA) and PARAFAC2. IVA [2] is an extension of Independent Component Analysis (ICA) to multiple datasets, and a good candidate for data fusion because it makes use of the dependence across datasets. The PARAFAC2 model [3] also has proved useful for jointly analyzing datasets as a more flexible version of the well-known CANDECOMP/PARAFAC tensor method.

With the goal of identifying novel biomarkers for complex neurological disorders, fusion of medical imaging data has received particular attention. Especially important is multi-task functional Magnetic Resonance Imaging (fMRI) data, i.e., data collected from the same subjects while they are performing different tasks. Since different tasks provide complementary information about the brain, analyzing the joint information between tasks may help to better understand these disorders.

In this talk, we study IVA and PARAFAC2 for data fusion [4], first through simulations, where multiple datasets in the form of *subjects* by *voxels* matrices correspond to different tasks. Our simulations reveal that both methods can accurately capture the underlying latent components, albeit with certain differences in capturing the corresponding subject scores. We then apply both methods for the analysis of 13 fMRI datasets from the MCIC collection [5], collected from 271 subjects that perform 3 different tasks with well-defined relationship among them. Both methods are able to achieve two important goals at once, namely capturing group differences between patients with schizophrenia and healthy controls with interpretable components, as well as understanding the relationship across the tasks.

This is joint work with Evrim Acar (Simulamet, Norway), Tanuj Hasija (Paderborn University), M.A.B.S. Akhonda (UMBC), Vince D. Calhoun (TReNDS Center), Peter J. Schreier (Paderborn University), and Tülay Adali (UMBC).

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Early soft and flexible fusion of EEG and fMRI via tensor decompositions for multi-subject group analysis

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Data fusion refers to the joint analysis of multiple datasets that provide different (e.g., complementary) views of the same task. In general, it can extract more information than separate analyses can. Jointly analyzing EEG and fMRI measurements has been proved to be highly beneficial to the study of the brain function, mainly because these neuroimaging modalities have complementary spatio-temporal resolution[1]: EEG offers good temporal resolution while fMRI is better in its spatial resolution. The EEG-fMRI fusion methods that have been reported so far ignore the underlying multi-way nature of the data in at least one of the modalities and/or rely on very strong assumptions concerning the relation of the respective datasets. For example, in multi-subject analysis it is commonly assumed that the Haemodynamic Response Function (HRF) is a-priori known for all subjects and/or the coupling across corresponding modes is assumed to be exact (hard). In this paper, these two limitations are overcome by adopting tensor models for both modalities and by following soft[2] (i.e., not hard) and flexible (i.e., possibly varying HRFs based on preselected family of models)[3] coupling approaches to implement the multi-modal fusion. The obtained results are compared against those of parallel Independent Component Analysis (ICA) and hard coupling alternatives, with both synthetic and real data (epilepsy and visual oddball paradigm). Our results demonstrate the clear advantage of using soft and flexible coupled tensor decompositions in scenarios that do not conform with the hard coupling assumption.

This is joint work with Eleftherios Kofidis (University of Piraeus), Simon Van Eyndhovem (Icometrix), Wim Van Paesschen (UZ Leuven), Lieven De Lathauwer (KU Leuven), Sergios Theodoridis (Aalborg University) and Sabine Van Huffel (KU Leuven). Supported by the European Union's 7^{th} *Framework Program under the ERC Advanced Grant: BIOTENSORS (n°* 339804*).*

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Parameter Estimation of Parallel Wiener-Hammerstein Systems by Decoupling their Volterra Representations

Mariya Ishteva

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Nonlinear dynamic systems are often modelled by a Volterra series (a generalization of the Taylor series). Unfortunately, the Volterra series lacks physical interpretation. To take advantage of the Volterra representation while aiming for an interpretable block-oriented model, we establish a link between the Volterra representation and the parallel Wiener-Hammerstein model. This link is based on decoupling of multivariate polynomials with (block-)Toeplitz structure on the factors and sets of identical internal branches.

The solution of the constrained decoupling problem reveals directly the parameters of the parallel Wiener-Hammerstein model of the system. However, imposing these constraints requires significant modification of the decoupling problem. Luckily, due to the uniqueness properties of the plain decoupling algorithm, even if the structure is not imposed, the method still leads to the true solution (in the exact case).

This is joint work with Philippe Dreesen (KU Leuven). Supported by KU Leuven Research Fund; FWO (EOS Project 30468160 (SeLMA), SBO project S005319N, Infrastructure project I013218N, TBM Project T001919N, G028015N, G090117N, SB/1SA1319N, SB/1S93918, SB/151622); Flemish Government (AI Research Program); European Research Council under the European Union's Horizon 2020 research and innovation programme (ERC AdG grant 885682), KU Leuven start-up-grant STG/19/036 ZKD7924. PD is affiliated to Leuven.AI - KU Leuven institute for AI, Leuven, Belgium.

Existence of best low rank approximations of positive definite tensors

Eric Evert

KU Leuven

Tensors, or multiindexed arrays, play an important role in fields such as machine learning and signal processing. These higher-order generalizations of matrices allow for preservation of higher-order structure present in data, and low rank decompositions of tensors allow for recovery of underlying information. In many cases, e.g., in blind source separation or diffusion tensor imaging, the underlying tensor of interest is positive (semi)definite. That is, the homogeneous polynomial associated to the tensor has nonnegative evaluation on all inputs.

An archetypal problem is that one has a noisy measurement of some low rank signal tensor of interest. This measurement itself does not have low rank, so one must compute a best low rank CPD approximation of the measured tensor. As it turns out, the set of tensors of rank less than or equal to R is in general not closed when $R > 1$, and, as a consequence, best low rank approximations can fail to exist. In the case that a best low rank approximation does not exist, near optimal low rank approximations suffer numerical issues and cannot be used to reliably approximate underlying component information. As a consequence, existence guarantees for best low rank tensor approximations are of great practical and theoretical interest.

This talk will give deterministic guarantees for the existence best low rank approximations of tensors which are positive semidefinite. In particular, we show that the set of low rank positive semidefinite tensors is relatively closed as a subset of the set of positive semidefinite tensors. We use this fact to give a deterministic bound which may be used to guarantee the existence of a best low rank approximation of a noisy low rank positive semidefinite tensor. In addition, for order three tensors, we prove that our bound is sharp and that it can be computed using semidefinite programming.

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Tensor-based methods for large-scale inverse problems in machine learning

Kim Batselier

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Large-scale linear inverse problems appear in myriads of applications ranging from astronomy to medicine[1, 2, 3, 4]. In this talk I will address a large-scale inverse problem that is known in the machine learning community as learning a kernel machine[5]. The same problem is also known in the control engineering community as nonlinear system identification [6]. First, I will discuss the forward model and highlight its inherent tensorial structure. This tensorial structure is the result from building multivariate basis functions as tensor products of univariate basis functions. Then, I will discuss how this structure can be exploited using tensor decompositions to enable efficient solving of large-scale inverse problems. The power of these tensor-based methods will be demonstrated in a live demo, where I will invert a dense square matrix of order 67 million on a standard laptop in about 5 seconds.

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Multilinear Modeling for Control and Diagnosis

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Many engineering applications in the area of controller design and fault diagnosis are based on linear models. This enables the use of efficient linear algebra algorithms as long as the systems behavior is - at least approximative - linear. But if larger deviations from operating points have to be considered, the engineering goals might not be achievable by linear models and the corresponding methods.

Nonlinear models are in principle able to overcome this. But they come in general with severe drawbacks due to their complexity: e.g. higher and unpredictable computation times as well as nonconvexity of underlying optimization problems. Moreover, model representations depend on the tools of information technology and not on abstract mathematical standards as the system, input, output and feedthrough matrices in linear system theory. The latter is mainly caused by the requirement that all kinds of nonlinearities should be representable.

But, as the system dynamics in several application areas as HVAC systems or power networks are intrinsically multilinear or can be multilinearized, their behaviour can be described appropriately by multilinear models, [1, 2]. Moreover, the parameters of multilinear time-invariant (MTI) models are tensors - which naturally extend parameter matrices of state space models for linear time-invariant (LTI) models. Explict multilinear models can be represented by full or decomposed tensors, which enable standardized formats finally leading to efficient as well as generic algorithms for simulation, controller design, and fault diagnosis.

One of the main obstacles of the explict MTI modeling approach is the non-closedness of the MTI class w.r.t. series and feedback compositions. Recent results will be presented, which show how this can be overcome by implict MTI models [3]. The usage of these models, e.g. at the Fraunhofer Application Center for Integration of Local Energy Systems (ILES) for large scale energy systems will be presented.

This is joint work with Georg Pangalos, Leona Schnelle, Carlos Cateriano Yáñez, Aline Luxa, Torben Warnecke, Niklas Jöres, Christoph Kaufmann and Aadithyan Sridharan supported by grants 13FH144PA8, 13FH1I05IA and 01LY1812B from the Federal Ministry of Education and Research Germany as well as the Free and Hanseatic city of Hamburg.

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Decoupling multivariate functions using a nonparametric filtered tensor decomposition

Jan Decuyper

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This work deals with the problem of function decoupling. Function decoupling is the process of approximating a multivariate real function $\boldsymbol{f}(\boldsymbol{p}) : \mathbb{R}^m \mapsto \mathbb{R}^n$, e.g. a neural network or multivariate polynomial, with a so-called decoupled form, $\boldsymbol{f}_d(\boldsymbol{p}) := \boldsymbol{W}\boldsymbol{g}\left(\boldsymbol{V}^\top \boldsymbol{p}\right)$, i.e. an additive structure built up out of univariate functions $g_i(z_i) : \mathbb{R} \mapsto \mathbb{R}$ of linear forms $z_i := \boldsymbol{v}_i^\top \boldsymbol{p}$, with $i = 1, \ldots, r$. The decoupled representation is often preferred given that it may result in a more efficient parameterisation, while additionally favouring explainability through the use of tractable univariate functions g_i. The functions g_i may be seen as tailored activation functions, characteristic of the underlying relationship. The approximation problem is solved using a numerical approach. The objective then assumes the form:

$$\underset{\boldsymbol{W},\boldsymbol{V},g_i \in \mathcal{G}}{\arg\min} \frac{1}{N} \sum_{k=1}^{N} \|\boldsymbol{f}(\boldsymbol{p}(k)) - \boldsymbol{f}_d(k)\|_2^2 \,, \tag{9}$$

i.e. a distance measure based on a selection of operating points $\{\boldsymbol{p}(k)\}_{k=1}^{N}$, and with $\mathcal{G}$ a predefined function family. A direct approach to (9) would, however, result in a hard nonlinear optimisation problem. Moreover, it would require predefining the function family $\mathcal{G}$, which may be non-trivial. These issues are mitigated in this presentation by introducing the process of decoupling based on filtered tensor decomposition [1]. Reformulating the objective at the level of the Jacobian allows framing the problem as a diagonal tensor decomposition [2], resulting in a number of advantages: 1) Exploiting a smoothness promoting penalty term, based on finite difference filters, enables retrieving meaningful decoupled functions, irrespective of the uniqueness properties of the decomposition. 2) An additional advantage is that the procedure is nonparametric, meaning that no a priori assumptions on the functional family of $\boldsymbol{g}$ are required. The method finds direct applications in machine learning where it can be used to increase the explainability of algorithms, e.g. by shedding light on the decision logic. Additionally, also the field of nonlinear system identification can benefit from decoupling techniques. Nonlinear black-box models are known to suffer from large numbers of parameters, while providing little to no insight into the underlying relationship. Decoupling can in those cases be used as a tool to retrieve parts of the underlying physics, as was demonstrated in [3].

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Tensor-based training of neural networks

Patrick Gelss

Freie Universität Berlin

The interest in machine learning with tensor networks has been growing rapidly in the past years. In this talk, we will discuss recently proposed tensor-based approaches for learning governing equations and image classification such as MANDy [1] and ARR [2]. The insights gained from these methods are used to develop a novel approach for training shallow neural networks. We show how the functional tensor-train format and Tikhonov regularization can be used to approximate solutions of Fredholm integral equations which describe infinitely large hidden layers. The efficiency and reliability of the introduced approach is illustrated with the aid of numerical experiments.

This is joint work with Aizhan Issagali, Carsten Gräser, and Ralf Kornhuber (FU Berlin).

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MS-22: Coding Theory and Linear Algebra over Finite Fields

Organisers: E. Byrne (UCD), A. Ravagnani (TU/e), J. Sheekey (UCD)

Theme: The focus of the symposium is on the interplay between linear algebra and coding theory.

The dual code of points and lines in a projective plane

Geertrui Van de Voorde

University of Canterbury

The *code* $C(\Pi)$ of points and lines in a projective plane Π of order q, $q = p^h$, p prime, is the $\mathbb{F}_p$-vector space generated by incidence matrix of points versus lines. The parameters of this code have been studied since the 1970's, in particular, the minimum weight is known, and the dimension is known in the Desarguesian case.

The *dual* code $C(\Pi)^\perp$ is the orthogonal complement (with respect to the standard dot product on $\mathbb{F}_p$) of the code $C(\Pi)$. Its minimum weight is not known in general. Even in the Desarguesian case, only the cases q even and q prime have been tackled.

In this talk, we will focus on the case $q = p^2$ and link codewords of certain small weights to the existence of embedded subplanes and *antipodal* planes. In the Desarguesian case we derive a non-embeddability results. Together with more combinatorial arguments, this allows us the improve on the currently best known lower bound for the minimum weight.

This is joint work with Maarten De Boeck (University of Rijeka, Croatia). Supported by the Marsden Fund Council, administered by the Royal Society of New Zealand, Grant MFP-UOC1805.

Cyclic line-spreads and flag-transitive linear Spaces

Cian Jameson

University College Dublin

There has been much progress towards classifying linear spaces that have a flag-transitive automorphism group in recent decades. However, a complete classification is not available as the case in which the automorphism group is a subgroup of one-dimensional affine transformations remains open.

In a 2007 paper [1], Pauley and Bamberg constructed flag-transitive linear spaces that lie in the open case via spreads and provided a condition for such spreads to exist. This condition links the existence of these flag-transitive linear spaces with polynomials.

In this talk, I will present my work on describing the polynomials that give rise to the desired linear spaces. I will provide conditions for binomials and cubic polynomials, as well as some results on equivalence of the resulting linear spaces.

This is joint work with my PhD supervisor John Sheekey (University College Dublin).

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Codes over finite fields and Galois ring valued quadratic forms

Ignacio F. Rúa

University of Oviedo (Spain)

In this talk we will address the construction of codes over finite fields from Galois ring valued quadratic forms.

This is joint work with Alejandro Piñera Nicolás (University of Oviedo). Supported by the Spanish MINECO, under Grant MTM-2017-83506-C2- 2-P.

Linearized Polynomials and Galois Groups

Gary McGuire

University College Dublin

Linearized polynomials have many applications in coding theory. We will discuss the Galois group G of a linearized polynomial over a field F of characteristic p, considered as a subgroup of $GL_n(F)$. This leads naturally to some representations of G over $\mathbb{F}_p$, and we mention some of the $\mathbb{F}_pG$ modules that arise. We also discuss Galois groups of projective polynomials.

This is joint work with Rod Gow (UCD).

Dynamic Proofs of Retrievability and Verified Evaluation of Secret Dotproducts and Polynomials

Jean-Guillaume Dumas

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We consider the problem of efficiently evaluating a secret polynomial at a given public point over a finite field, when the polynomial is stored on an untrusted server.

The server performs the evaluation and returns a certificate, and the client can efficiently check that the evaluation is correct using some pre-computed keys. Our protocols support two important features: the polynomial itself can be encrypted on the server, and it can be dynamically updated by changing individual coefficients cheaply without redoing the entire setup. Our technique can also be used similarly for the verified computation of a dotproduct by a server where one of the vector remains secret.

Our methods rely only on linearly homomorphic encryption and pairings. Still we provide experiments showing that our client verification can be orders of magnitude faster than re-computation.

As an important application, we show how these new techniques can be used to instantiate a Dynamic Proof of Retrievability (DPoR) for arbitrary outsourced data storage that achieves both low server storage size and audit complexity. Indeed, PoRs are protocols which allow a client to store data remotely and to efficiently ensure, via audits, that the entirety of that data is still intact. A *dynamic* PoR system also supports efficient retrieval and update of any small portion of the data.

We propose new and simple protocols for dynamic PoR where the audits are based on verifiable linear and polynomial algebra computations over finite fields. Again, our protocols are designed for practical efficiency, trading decreased persistent storage for increased server computation. They are the first dynamic PoR which do not require any special encoding of the data stored on the server, meaning it can be trivially composed with any database service or with existing techniques for encryption or redundancy. We also present several further enhancements, reducing the amount of client storage, or the communication bandwidth, or allowing *public verifiability*, wherein any untrusted third party may conduct an audit.

Our implementation and deployment on Google Cloud Platform demonstrates our solution is scalable: for example, auditing a 1TB file takes just less than 5 minutes and costs less than \$0.08 USD.

This is joint work with Aude Maignan (Univ. Grenoble Alpes), Clément Pernet (Univ. Grenoble Alpes) and Daniel S. Roche (US Naval academy).

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One-Shot Capacity of Networks with Restricted Adversaries

Altan Berdan Kiliç

Eindhoven University of Technology

In this talk, we will concentrate on the one-shot capacity of communication networks with an adversary who can possibly corrupt only a proper subset of network edges. That is, we are interested in computing the maximum number of information symbols that can be sent in a single use of the network, no matter how the adversary acts. We show that linear network coding does not suffice in general to achieve capacity, proving a strong separation result between the one-shot capacity and its linear version and contrasting this with the classical network coding setting where the adversary is not restricted. We then give a general method to obtain upper bounds on the said capacity by studying some induced networks with only two levels of vertices.

This is joint work with Allison Beemer and Alberto Ravagnani. Supported by the Dutch Research Council, Grant VI.Vidi.203.045.

On Cameron-Liebler sets in projective spaces, and low degree Boolean functions

Jan De Beule

Vrije Universiteit Brussel

Let $\mathrm{PG}(n, q)$ denote the n-dimensional projective space over the finite field $\mathbb{F}_q$. We assume $n \geq 3$. Let $0 \leq d < k < n$, and let A be the d-space-k-space incidence matrix, i.e. the rows of $A = (a_{ij})$ are indexed by the d-dimensional subspaces of $\mathrm{PG}(n, q)$, the columns by the k-dimensional subspaces $\mathrm{PG}(n, q)$ and $a_{\pi,\sigma} = 1$ if and only if $\pi \subset \sigma$ and $a_{\pi,\sigma} = 0$ otherwise.

Let $d = 0$. A *Cameron-Liebler set of k-spaces* is a set $\mathcal{C}$ of k-spaces such that the characteristic vector $\chi_{\mathcal{C}} \in \mathrm{Im}(A^T)$. These objects are natural generalizations of Cameron-Liebler line classes ($k = 1$ in the definition), which where introduced by Cameron and Liebler to study irreducible collineation groups in $\mathrm{PG}(n, q)$ having equally many orbits on the points as on the lines.

These objects are well studied in their geometrical context. In this talk, first we summarize old and recent results, including equivalent characterizations, non-existence conditions, and non-trivial examples, all for $k = 1$. Then we present recently obtained existence conditions for $k > 1$. Finally we also discuss the connection between Cameron-Liebler sets of k-spaces and Boolean functions of degree 1, and a geometrical approach to construct Boolean degree functions of low degree $d > 1$.

This is joint work with Jozefien D'haeseleer (Ghent University), Ferdinand Ihringer (Ghent University), Jonathan Mannaert (Vrije Universiteit Brussel), and Leo Storme (Ghent University)

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Trifferent codes and affine blocking sets

Anurag Bishnoi

TU Delft

Trifferent codes, also known as perfect 3-hash codes, are subsets of C of $\{0,1,2\}^n$ such that for any three distinct codewords in C, there is a common coordinate position where all of these codewords have different values. When $\{0,1,2\}$ is identified with $\mathbb{F}_3$ and C is a linear subspace of $\mathbb{F}_3^n$, then it is called a linear trifferent code. Studying the maximum possible size of trifferent codes of length n, as a function of n, is one of the classic open problems in both coding theory and extremal combinatorics. The trivial upper bound of $c\left(\frac{3}{2}\right)^n$ has not been improved despite considerable effort, except for improvements in the constant c. The best known lower bound is also exponential but with a smaller base of the exponent. Recently, Pohoata and Zakharaov studied linear trifferent codes and showed a much stronger upper bound on their size, compared to trifferent codes. In this talk we will present further improvements to their upper bound and new exponential lower bounds. We also propose a natural problem in finite geometry, where explicit constructions can potentially lead to the best known explicit lower bounds on (not necessarily linear) trifferent codes.

This is joint work with Dion Gijswijt, Jozefien D'haesleer and Aditya Potukuchi.

Independent Spaces of q-Polymatroids

Heide Gluesing-Luerssen

University of Kentucky

It is well known that $\mathbb{F}_{q^m}$-linear rank-metric codes in $\mathbb{F}_{q^m}^n$ give rise to q-matroids while the more general $\mathbb{F}_q$-linear rank-metric codes in $\mathbb{F}_q^{n\times m}$ lead to q-polymatroids [4, 5]. The latter differ from q-matroids in that the rank function may assume rational values. Just like for (classical) matroids and polymatroids, this generality of the rank function has vast consequences for the theory of q-polymatroids. While for q-matroids a variety of cryptomorphic descriptions have been established [1], little is known so far for q-polymatroids.

In this talk we introduce, for any common denominator μ of the rank function, a notion of μ-independent spaces for q-polymatroids. With the aid of an auxiliary q-matroid, we establish properties of the collection of μ-independent spaces that resemble those for q-matroids. This allows us to show that the entire q-polymatroid is fully determined by the collection of μ-independent spaces along with their rank values, and one arrives at a cryptomorphism of q-polymatroids based on independent spaces. Examples show that no such cryptomorphism is possible using only bases, dependent spaces, or circuits (along with their rank values). This is based on joint work with Benjamin Jany [2, 3].

Supported by Simons Foundation Grant 422479.

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Rank-Metric Lattices

Giuseppe Cotardo

University College Dublin

Higher-Weight Dowling Lattices (HWDL in short) are special families of geometric lattices introduced by Dowling [3] in connection with coding theory. These lattices were further studied, among others, by Bonin [1, 2], Kung [4], and more recently by Ravagnani [5]. The elements of HWDLs are the $\mathbb{F}_q$-linear subspaces of $\mathbb{F}_q^n$ having a basis of vectors with Hamming weight bounded from above, ordered by inclusion.

In this talk, we define and investigate structural properties of the q-analogues of HWDLs, which we call rank-metric lattices (RML in short). Their elements are the $\mathbb{F}_{q^m}$-linear subspaces of $\mathbb{F}_{q^m}^n$ having a basis of vectors with rank weight bounded from above, ordered by inclusion. We determine which RMLs are supersolvable, computing their characteristic polynomials. In the second part of the talk, we establish a connection between RMLs and the problem of distinguishing between inequivalent rank-metric codes.

The new results in this talk are joint work with A. Ravagnani (Eindhoven University of Technology). Supported by the Irish Research Council, grant n. GOIPG/2018/2534.

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MRD Codes and the Average Critical Problem

Anina Gruica

Eindhoven University of Technology

This talk will be about two problems intersecting coding theory and combinatorial geometry, where the focus lies on their relationship. These are the problem of computing the asymptotic density of MRD codes in the rank metric, and the Critical Problem by Crapo and Rota.

While it is known that MRD codes are generally sparse within the set of codes of the same dimension for q large, computing the exact asymptotic behavior of their density is still a wide open question. A natural step towards solving this problem is to obtain lower bounds on their number. I will show how the theory of semifields can be used to get a lower bound for the number of full-rank, square MRD codes. This lower bound is tight when n is prime and q is large, which gives a closed formula for their density function.

In the second part of the talk, I will focus on the Critical Problem for combinatorial geometries, approaching it from a different (more qualitative, often asymptotic) viewpoint. Finally, I will present the connection between this very classical problem and that of computing the asymptotic density of MRD codes.

This is joint work with Alberto Ravagnani, John Sheekey and Ferdinando Zullo. My research is supported by the Dutch Research Council through grant OCENW.KLEIN.539.

From linear to non-linear functions over finite fields

Ferdinando Zullo

Università degli Studi della Campania "Luigi Vanvitelli"

Let $\mathbb{F}_{2^n}$ be the finite field with 2^n elements. Given a function $f : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$, it is interesting to understand how many solutions x the equation

$$f(x+a)+f(x)=b \tag{10}$$

has, for any $a \in \mathbb{F}_{2^n} \setminus \{0\}$ and $b \in \mathbb{F}_{2^n}$. A function f is said to be **almost perfect nonlinear** (APN) if there are always exactly zero or two solutions to (10).

APN functions were introduced by Nyberg in 1993, in the context of cryptography, as the mappings with highest resistance to differential cryptanalysis. Since then they appeared in several contexts, such as for the construction of semi-biplanes, dual-hyperovals and linear codes.

In this talk we consider the new family of quadratic APN functions, recently introduced in [2], that generalizes the one by Bracken, Byrne, Markin and McGuire in 2011. Let s and m be integers such that $\gcd(s,m)=1$. The mapping defined over $\mathbb{F}_{2^{3m}}$

$$(x^{2^{m+s}}+\mu x^{2^s}+x)^{2^m+1}+vx^{2^m+1}, \tag{11}$$

where $\mu \in \mathbb{F}_{2^{3m}}$ satisfies $\mathrm{N}_{2^{3m}/2^m}(\mu) := \mu^{2^{2m}+2^m+1} \neq 1$ and $v \in \mathbb{F}_{2^m}^*$, is APN whenever $f_\mu^{(s)}(x) := x^{2^{m+s}}+\mu x^{2^s}+x$ permutes $\mathbb{F}_{2^{3m}}$.

In [1], we proved the existence for all $m \geq 3$ of suitable s and μ, $\mathrm{N}_{2^{3m}/2^m}(\mu) \neq 1$, for which the polynomial (11) is APN. A key tool in our machinery is the investigation of the kernel of 2-linearized polynomials (that is $\mathbb{F}_2$-linear maps of $\mathbb{F}_{2^{3m}}$) of the type $f_\mu^{(s)}(x) := x^{2^{m+s}}+\mu x^{2^s}+x \in \mathbb{F}_{2^{3m}}[x]$, which belong to a family of linearized polynomials already investigated in [3].

This is joint work with Daniele Bartoli (Università degli Studi di Perugia), Marco Calderini (Università degli Studi di Trento) and Olga Polverino (Università degli Studi della Campania "Luigi Vanvitelli"). Supported by the project "VALERE: VAnviteLli pEr la RicErca" of the University of Campania "Luigi Vanvitelli" and was partially supported by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INdAM).

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Contributed Sessions

Organisers: Helena Šmigoc and Oliver Mason

Theme: The talks in the contributed sessions demonstrate the depth and breath of research by members of the linear algebra community. They present novel contributions to the field, and demonstrate the interplay between linear algebra and many other disciplines, including data science, network analysis, epidemiology, and a host of other fields.

Monday, 20 June

Contributed Session 1

Contributed Session 2A

Contributed Session 2B

Tuesday, 21 June

Contributed Session 3A

21 June	10:30	AC203	**Ivana Šain Glibić**	p260
Importance of the deflation process for the solution of quartic eigenvalue problem				
21 June	11:00	AC203	**Avleen Kaur**	p261
How the Friedrichs angle leads to lower bounds on the minimum singular value				
21 June	11:30	AC203	**George Hutchinson**	p263
On the enumeration and properties of complex matrix scalings				

Contributed Session 3B

21 June	10:30	AC204	**Dmitry Savostyanov**	p259
Tensor product approach to epidemiological models on networks				
21 June	11:00	AC204	**Ryan Wood**	p262
Dynamic Katz and Related Network Measures				
21 June	11:30	AC204	**Cheolwon Heo**	p264
The Complexity of the Matroid-homomorphism problems				
21 June	12:00	AC204	**Sophia Keip**	p265
Kirchberger's Theorem for Complexes of Oriented Matroids				

Contributed Session 4A

21 June	14:00	AC203	**Plamen Koev**	p266
Accurate Bidiagonal Decompositions of Structured Totally Nonnegative Matrices with Repeated Nodes				
21 June	14:30	AC203	**Michael Tsatsomeros**	p269
The Fiber of P-matrices: the Recursive Construction of All Matrices with Positive Principal Minors				
21 June	15:00	AC203	**Raquel Viaña**	p270
Accurate computation of the inverse of Totally Positive collocation matrices of the Lupaş-type (...				
21 June	15:30	AC203	**Adi Niv**	p272
Tropical Matrix Identities				

Contributed Session 4B

21 June	14:00	AC204	**Lauri Nyman**	p267
Perturbation theory of transfer function matrices				
21 June	14:30	AC204	**Patricia Antunes**	p268
Bi-additive Models: different types of distributions				
21 June	15:00	AC204	**Juyoung Jeong**	p271
Weak majorization inequalities in Euclidean Jordan algebras				
21 June	15:30	AC204	**Luis Felipe Prieto-Martínez**	p273
Geometric continuity, Riordan matrices and applications				

Wednesday, 22 June

Contributed Session 5A

Contributed Session 5B

Thursday, 23 June

Contributed Session 6A

23 June	10:30	AC204	**Riadh ZORGATI**	p281
Projections, L_p Norms and Stochastic Matrices for Ill-Conditioned Linear Systems of Equations				
23 June	11:00	AC204	**Philippe Dreesen**	p283
Solving (Overdetermined) Polynomial Equations				
23 June	11:30	AC204	**Eric de Sturler**	p285
Efficient Computation of Parametric Reduced Order Models using Randomization				
23 June	12:00	AC204	**Alicia Roca**	p286
The change of the Weierstrass structure under one row perturbation				

Contributed Session 6B

23 June	10:30	Anderson	**André Ran**	p280
Rational matrix solutions to $p(X) = A$				
23 June	11:00	Anderson	**Héctor Orera**	p282
Bidiagonal decomposition and accurate computations with matrices of q-integers				
23 June	11:30	Anderson	**Yinfeng Zhu**	p284
Hurwitz primitivity and synchronizing automata				
23 June	12:00	Anderson	**Raf Vandebril**	p287
Construction of a sequence of orthogonal rational functions				

Contributed Session 7

23 June	14:00	AC215	**Madelein van Straaten**	p288
H-selfadjoint mth roots of H-selfadjoint matrices over the quaternions				
23 June	14:30	AC215	**Dawie Janse van Rensburg**	p289
An alternative canonical form for quaternionic H-unitary matrices.				
23 June	15:00	AC215	**M. Eulàlia Montoro**	p290
The combinatory under isomorphic lattices of hyperinvariant subspaces				

Blocked Triangular Patterns and their Symmetry Groups

James R. Weaver

University of West Florida

A $2n \times 2n$ matrix partitioned into $n \times n$ blocks is called a triangular pattern if the entry pattern of each block is one of the triangles determined by the main diagonal or anti-diagonal. The dihedral group of order 8, $D_4(8)$, realized as a subgroup of the group S_{2n} of $2n \times 2n$ blocked permutation matrices, acts via conjugation on the set of triangular patterns Δ. Patterns P and Q in Δ are $D_4(8)$-equivalent if there is a permutation $\Phi \in D_4(8)$ such that $\Phi P \Phi^T = Q$. The objective of this paper is to examine the action of $D_4(8)$ on Δ. Of particular interest are the orbits of this group action, and certain other subgroups of S_{2n} associated with $D_4(8)$.

This is joint work with James E. Brewer and Rohan Hemasinha.

Majorization and Triangular Polynomial Matrices

Richard Hollister

University at Buffalo

Majorization is a partial ordering of $\mathbb{R}^n$ with numerous applications across many areas of study, [2, 4]. This classical concept can trace its origins back to an equivalent ordering given by Muirhead in 1903, [5]. In this talk, we discuss recent work highlighting the connection between majorization and the diagonals of triangular matrix polynomials. The question being addressed was first considered in the PhD thesis of Eduardo Marques de Sá, [3]: which polynomial diagonals are possible in a triangular realization of a given Smith form? The current work answers the same question, but in a conceptually and computationally simpler way using majorization. The end result is an implementable algorithm that can be used to compute such a triangular realization.

This work relates to research investigating the triangularization of matrix polynomials. It was shown in [6] that every regular matrix polynomial over an algebaically closed field can be triangularized. In more recent work by Anguas, Dopico, Hollister, and Mackey, it was shown that every regular matrix over an *arbitrary field* can be quasi-triangularized with diagonal blocks having bounded sizes, [1]. The role that majorization plays in these results is also discussed.

This is joint work with Luis Miguel Anguas (Universidad Politécnica de Madrid), Froilán Dopico (Universidad Carlos III de Madrid), and D. Steven Mackey (Western Michigan University). Supported by "Proyecto de I+D+i PID2019-106362GB-I00 financiado por MCIN/AEI/10.13039/501100011033" and "Ministerio de Economía, Industria y Competitividad (MINECO)" of Spain through grants MTM-2015-65798-P and BES-2013-065688.

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Spectral Localization in Polynomial and Rational Matrices

D. Steven Mackey

Western Michigan University

Let $P(\lambda)$ be any polynomial matrix over an arbitrary field $\mathbb{F}$, and $q(\lambda)$ a monic irreducible scalar polynomial over $\mathbb{F}$. Define $\mathcal{L}oc_q(P)$, the "localization of P at q", to be the polynomial matrix with the same zero entries as $P(\lambda)$, but with each nonzero entry $p_{ij}(\lambda)$ of $P(\lambda)$ replaced by a power of q, in particular by $q^{e_{ij}}(\lambda)$ where $p_{ij}(\lambda) = q^{e_{ij}}(\lambda)r(\lambda)$ with r coprime to q, and $e_{ij} \geq 0$. Note that if $\mathsf{Smith}(P)$ is the Smith form of P, then $\mathcal{L}oc_q\big(\mathsf{Smith}(P)\big)$ is often referred to as the "local Smith form of P at q", since it displays all of the elementary divisors of P at q, and nothing else about the spectral structure of P.

Now for anything other than diagonal matrices, it would at first sight seem crazy to think that the elementary divisors of the drastically simplified matrix $\mathcal{L}oc_q(P)$ would have anything at all to do with the elementary divisors of P at q, let alone be exactly the same. In other words, to think that

$$\mathsf{Smith}\big(\mathcal{L}oc_q(P)\big) \;=\; \mathcal{L}oc_q\big(\mathsf{Smith}(P)\big) \tag{12}$$

could possibly be true. But there is in fact a substantial class of non-diagonal polynomial matrices P for which the relation (12) does hold, indeed holds for *all* monic irreducible q. For example, all bidiagonal polynomial matrices have this property, which I will refer to as the *spectral localization property.*

This talk will discuss some of the basic results concerning the spectral localization property, and describe how to build many examples of polynomial matrices that possess it. As time permits, I will also indicate how these results can be used to help design polynomial matrices with specified finite and infinite elementary divisors, and how to extend these results to rational matrices, using the Smith-McMillan form in place of the Smith form.

Circular higher rank numerical range and factorization of singular matrix polynomials

Edward Poon

Embry-Riddle Aeronautical University

Abstract: The rank-k numerical range of a square matrix A is the set of complex numbers λ for which there exists an orthogonal rank-k projection P such that $PAP = \lambda P$. We present conditions which guarantee that such a higher rank numerical range is a circular disk. Our results generalize Anderson's Theorem [1], and are in turn generalized to provide factorizations of singular Hermitian-valued trigonometric polynomials on the unit circle.

This is joint work with Ilya Spitkovsky (NYUAD) and Hugo Woerdeman (Drexel).

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New Connections between Static Matrices A, Zhang Neural Networks, and Parameter-Varying Matrix Flows $A(t)$

Frank Uhlig

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We study recent results that connect classical matrix theory with matrix computations in new ways.

Field of values computations lead me to study parameter-varying matrix eigenvalue problems. These concepts helped me solve a century old matrix block diagonalization problem that arose in Quantum Physics in the 1920s and was still open.

On the way we need to understand time-varying matrix problems theoretically and computationally when specifically approached through Zeroing Neural Networks.

Zhang Neural Networks (ZNN) are being used today in hundreds of ways in modern engineering, in robot control etc.

But their numerical behavior is barely understood at this time. ZNN methods pose many open problems and form a new, non-Wilkinsonian branch of Numerical Matrix Analysis.

In return, ZNN methods for matrix flows $A(t)$ can help us for the first time to solve long standing intractable computational problems of fixed entry matrices A such as the matrix symmetrizer problem.

A characterization of signed graphs with stable maximum nullity at most two

Marina Arav

Georgia State University

A signed graph is a pair (G, Σ) where G is an undirected graph (we allow parallel edges but no loops) and $\Sigma \subseteq E(G)$. If (G, Σ) is a signed graph with vertex-set $V = \{1, \dots, n\}$, $S(G, \Sigma)$ is the set of all $n \times n$ real symmetric matrices $A = [a_{i,j}]$ with $a_{i,j} > 0$ if i and j are adjacent and connected by only odd edges, $a_{i,j} < 0$ if i and j are adjacent and connected by only even edges, $a_{i,j} \in \mathbb{R}$ if i and j are adjacent and connected by both even and odd edges, $a_{i,j} = 0$ if i and j are not adjacent, and $a_{i,i} \in \mathbb{R}$ for all vertices i. The parameter $\xi(G, \Sigma)$ is defined as the largest nullity of any matrix $A \in S(G, \Sigma)$ satisfying the Strong Arnold Property. This invariant is closed under taking minors. In 2021, Arav, Hall, van der Holst, and Li gave a characterization of 2-connected signed graphs (G, Σ) with $\xi(G, \Sigma) \leq 2$. A full characterization was still open. In this talk, we discuss a full characterization of signed graphs (G, Σ) with $\xi(G, \Sigma) \leq 2$.

This is joint work with F. Scott Dahlgren (Georgia State University) and Hein van der Holst (Georgia State University).

Null-Space Projects for Intermediate Students: Tomography, Cryptography, and More

Tom Asaki

Washington State University

For many students, a non-trivial nullspace is a "necessary evil" associated with a non-injective linear transformation, $T : \mathbb{R}^n \to \mathbb{R}^m$. They understand that such transformations are not invertible, but strategies, such as least squares methods and pseudo-inversion via singular value decomposition, are designed to avoid nullspace contributions using approximate inverse transformations, $P : \mathbb{R}^m \to \mathbb{R}^n$. And, theorems even demonstrate that these methods represent best strategies, in that, recovered domain-space vectors are optimal. That is, if $T(x) = b$ and $\hat{x} = P(T(x))$, then $\|\hat{x} - x\|_2$ is minimized over all possible linear transformations P. In this work we describe improved pseudo-inversion methods that directly incorporate nullspace vectors along with both actual and introduced prior knowledge. We provide two examples suitable for curious students who wish more than nullspace avoidance. These examples provide a springboard for additional project directions.

One project concept is improved tomographic reconstructions from radiographs. The radiographic (approximate) linear transformation typically has a very large nullspace $n \gg m$. Any pseudo-inverse transformation typically results in a reconstruction which is unacceptable without extensive post-processing. We invite the student to use reasonable prior knowledge, such as accepting values from a given set, to find null vector contributions which enhance the result. In many cases, dramatic improvement is realized, up to and including exact reconstructions.

A second project concept is the sending and deciphering of encrypted messages. A message encrypted using a non-injective transformation is not simply recovered because of the loss of information. However, with the introduction of an intertwined and simultaneously encrypted passphrase, the correct nullspace contribution can be recovered. The transformation and encrypted message can be made public, while the private passphrase in known only to the sender and intended receiver.

A topological characterization of signed graphs with stable positive semidefinite maximum nullity at most two

Hein van der Holst

Georgia State University

A signed graph is a pair (G, Σ), where G is an undirected graph (we allow parallel edges but no loops) and $\Sigma \subseteq E(G)$. The edges in Σ are called odd, while the other edges are called even. If (G, Σ) is a signed graph with vertex-set $V = \{1, \ldots, n\}$, $S(G, \Sigma)$ is the set of all real symmetric $n \times n$ matrices $A = [a_{i,j}]$ with $a_{i,j} > 0$ if i and j are adjacent and connected by only odd edges, $a_{i,j} < 0$ if i and j are adjacent and connected by only even edges, $a_{i,j} \in \mathbb{R}$ if i and j are adjacent and connected by both even and odd edges, $a_{i,j} = 0$ if i and j are not adjacent, and $a_{i,i} \in \mathbb{R}$ for all vertices i. The parameter $\nu(G, \Sigma)$ is defined as the largest nullity of any positive semidefinite matrix $A \in S(G, \Sigma)$ satisfying the Strong Arnold Hypothesis. This invariant is closed under taking minors. Arav, Hall, van der Holst, and Li gave a forbidden minor characterization of the class of signed graphs (G, Σ) with $\nu(G, \Sigma) \leq 2$. In this talk we present a topological characterization of the class of signed graphs (G, Σ) with $\nu(G, \Sigma) \leq 2$.

Inverse of a signless Laplacian matrix of a non-bipartite graph

Milica Anđelić

Department of Mathematics, Kuwait University, Kuwait

We provide a relation between the Moore-Penrose inverse of the Laplacian and signless Laplacian matrices of a bipartite graph. As a consequence we present combinatorial formulae for the Moore-Penrose inverse of signless Laplacians of bipartite graphs. We also obtain a combinatorial formula for the Moore-Penrose inverse of an incidence matrix and derive a combinatorial formula for the inverse of signless Laplacians of non-bipartite graphs. These results answer some of the open problems raised in [R. Hessert, S. Mallik, Moore-Penrose inverses of the signless Laplacian and edge-Laplacian of graphs, Discrete Math. 344 (2021) #112451].

This is joint work with Abdullah Alazemi (Kuwait University) and Osama Alhalabi (Kuwait University), supported by the Research Sector, Kuwait University, Grant SM01/19.

Graph addition: properties for its use for graph protection

Vicenç Torra

Umeå University, Sweden

Graphs are useful to model complex systems (e.g., online social networks). Publishing a graph can lead to the disclosure of personal (or sensitive) information. Nodes may be identified under assumptions of different intensity: from knowing their degree distribution to being able to perform subgraph matching.

The noise-graph addition technique [2] is a way of protecting from such linkage. Let G_0 be a graph to protect, which can be represented by the adjacency matrix. Then, a random graph G from a given family is added to G_0, and this provides a new graph:

$$G' = G_0 \oplus G.$$

Here, addition $\oplus$ results into those edges in the symmetric difference of the edges of G and of G_0.

We can prove that this operation provides a metric. Additional properties on G' can be proven taking into account those from G_0 and of the random graph G.

Singular value decomposition (SVD) and nonnegative matrix factorization (NMF) have been used for community detection [3, 4]. The stochastic block model is a model of networks with community structure. They are tools useful for analyzing G_0, for generating random graphs G, and, thus, for creating G'. We study the effect of the noise-graphs in the SVD of the protected graphs and in relation to the stochastic block models.

This is joint work with Julian Salas (Open University of Catalonia). This study was partially funded by the Wallenberg AI, Autonomous Systems and Software Program (WASP) funded by the Knut and Alice Wallenberg Foundation.

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Tensor product approach to epidemiological models on networks

Dmitry Savostyanov

University of Essex, UK

Epidemiological modelling is crucial to inform healthcare policies and to support decision making for disease prevention and control. The recent outbrake of COVID-19 pandemic raised a significant scientific and public debate regarding the quality of the mathematical models used to predict the effect of the pandemics and to choose an appropriate response strategy. One of the first epidemiological models, proposed by Kormack and McKendrick in 1927, assumes that each member of the population, be it a susceptible, infected, or recovered person, has the same chance of getting in contact with other members. The assumption that the population is well–mixed simplifies the mathematical description of the model, but limits the accuracy of the results, because it ignores the information on where the infected people are located in relation to the susceptible part of the population.

In contrast, network–based models include information on how often people contact each other, hence providing a more realistic description of the population. Unfortunately, their complexity grows exponentially with the size of the network — these models suffer from the curse of dimensionality and usually rely on further approximations to make them practically solvable. In this talk we discuss how epidemiological models on networks can be solved accurately using the recently proposed algorithms based on low–rank tensor product factorisations. We demonstrate a few examples where the use of tensor product algorithms deliver more accurate results much faster than the Gillespie's stochastic simulation algorithm, widely used for this problem.

This is joint work with Sergey Dolgov (University of Bath, UK). This work is supported by the Leverhulme Trust Research Fellowship RF-2021-258.

Importance of the deflation process for the solution of quartic eigenvalue problem

Ivana Šain Glibić

University of Zagreb, Faculty of Science, Department of Mathematics

In this talk, we consider quartic eigenvalue problems, i.e. polynomial problems of degree 4. Our approach to numerical solution of this problem is to define corresponding linearization, and then compute eigenvalues of the obtained generalized eigenvalue problem. However, before going to QZ algorithm, we propose additional steps in order to improve the overall solution of the QZ algorithm.

The strong point of algorithm is the deflation of zero and infinite eigenvalues. The existence of these eigenvalues is determined by computing numerical rank of leading and constant coefficient matrices. Proposed procedure is based as much as possible on the initial data.

We will analyse the details of the deflation process for the quartic eigenvalue problem. By presenting carefully chosen numerical experiments, we will point out the importance and the overall influence of deflation on the final result.

This is joint work with Zlatko Drmač (University of Zagreb). Supported in part by the Croatian Science Foundation, UIP-2019-04-5200.

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How the Friedrichs angle leads to lower bounds on the minimum singular value

Avleen Kaur

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Estimating the eigenvalues of a sum of two symmetric matrices, say $P+Q$, in terms of the eigenvalues of P and Q, has a long tradition. To our knowledge, no study has yielded a positive lower bound on the minimum eigenvalue, $\lambda_{\min}(P+Q)$, when $P+Q$ is symmetric positive definite with P and Q singular positive semi-definite. We derive two new lower bounds on $\lambda_{\min}(P+Q)$ in terms of the minimum positive eigenvalues of P and Q. The bounds take into account geometric information by utilizing the Friedrichs angles between certain subspaces. The basic result is when P and Q are two non-zero singular positive semi-definite matrices such that $P+Q$ is non-singular, then $\lambda_{\min}(P+Q) \geq (1-\cos\theta_F)\min\{\lambda_{\min}(P), \lambda_{\min}(Q)\}$, where $\lambda_{\min}$ represents the minimum positive eigenvalue of the matrix, and θ_F is the Friedrichs angle between the range spaces of P and Q. We will discuss the interaction between the range spaces for some pair of small matrices to elucidate the geometric aspect of these bounds. Such estimates lead to new lower bounds on the minimum singular value of full rank 1×2, 2×1, and 2×2 block matrices in terms of the minimum positive singular value of these blocks. Some examples provided in this talk further highlight the simplicity of applying the results in comparison to some existing lower bounds.

This is joint work with S. H. Lui (Manitoba). Supported by the University of Manitoba Graduate Fellowship (Avleen Kaur) and the Natural Sciences and Engineering Research Council of Canada (S. H. Lui).

Dynamic Katz and Related Network Measures

Ryan Wood

Aalto University

The identification of important nodes within a network is a key feature of complex network analysis [1]. This is achieved using centrality measures, which are functions that assign a non-negative real value to each node in the network. These values induce a rank ordering of the nodes which is reflective of their relative importance within the network.

One significant class of centrality measures is the walk-based centrality measures, for which the value assigned to a given node is based on the number of walks which begin (or end) at that node. One well-known walk-based centrality is classical Katz centrality [2] and is given by the formula:

$$\boldsymbol{x}_{Katz}(t) = (I - tA)^{-1}e \tag{13}$$

where A is the adjacency matrix associated with the network, and e is the vector of all 1's of such a length that is coherent with A.

Non-backtracking Katz centrality is a variant of Katz centrality which discounts walks which involve a sequence of nodes of the form $u \to v \to u$. This offers several benefits.

Firstly, walks which backtrack in the manner described above can be unrealistic within the context of the network model. An archetypal example of such a context being instant messaging or email networks, in which it is unlikely that one reports received information back to its messenger.

Secondly, discounting non-backtracking walks is known to offer concrete benefits such as avoiding localisation in the eigenvectors of the non-backtracking adjacency matrix [3]. For temporal networks however, the enumeration of non-backtracking walks is further complicated by the possibility to backtrack not only spatially, but temporally also.

The subject of this talk will be the exposition of a multigraph approach which provides a combinatorially correct formula for non-backtracking centrality measures defined by analytic functions for temporal networks [4].

This is joint work with Arrigo, Francesca (University of Strathclyde), Higham, Desmond J. (University of Edinburgh) and Noferini, Vanni (Aalto University).

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On the enumeration and properties of complex matrix scalings

George Hutchinson

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The study of matrix scalings began in earnest with Richard Sinkhorn in 1964, and the subsequent decades produced many variations and generalizations of his original "classical" scaling.

In this talk, we will discuss one particular variation – the *complex matrix scaling*, introduced by Rajesh Pereira in 2003: Given an $n \times n$, positive definite (complex) matrix A and a diagonal matrix D, we say that D *scales* A if D^*AD has all row and column sums equal to 1. We will discuss recent progress made towards several open problems concerning the enumeration and properties of these scalings. We also give an application to the field of quantum information, using the permanent of these scalings to arrive at a bound on the geometric measure of entanglement of certain symmetric states.

Despite the application to quantum information, this talk is designed to be accessible to anyone familiar with elementary matrix theory.

The Complexity of the Matroid-homomorphism problems

Cheolwon Heo

Applied Algebra and Optimization Research Center, Sungkyunkwan University

In this talk, we introduce homomorphisms between binary matroids that generalize graph homomorphisms. For a binary matroid N, we prove a complexity dichotomy for the problem $\mathrm{Hom}_{\mathbb{M}}(\mathrm{N})$ of deciding if a binary matroid M admits a homomorphism to N. The problem is polynomial time solvable if N has a loop or has no circuits of odd length, and is otherwise NP-complete. We also get dichotomies for the list, extension, and retraction versions of the problem.

This is joint work with Hyobin Kim and Mark Siggers at Kyungpook National University.

Kirchberger's Theorem for Complexes of Oriented Matroids

Sophia Keip

FernUniversität in Hagen

This talk shows how a new abstract structure that only uses very few axioms can simplify the proof for an old, classical separation theorem, namely Kirchberger's Theorem.

Kirchberger's Theorem: Let V and W be finite subsets of $\mathbb{R}^n$. If every set $C \subseteq V \cup W$ of $n+2$ or fewer points can be strictly separated into the sets $V \cap C$ and $W \cap C$, then V can be strictly separated from W, i.e. one can find $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $a^T v - \alpha < 0$ for all $v \in V$ and $a^T w - \alpha > 0$ for all $w \in W$.

The original proof of Kirchberger in 1902 [3] is long and quite hard to read. Nowadays easier proofs are known. One possibility is to prove it using Helly's Theorem like in [2] or [4]. There is also a simpler proof that is basically a combination of Carathéodory's Theorem and Farkas' Lemma, which can be found in [5]. Since these two theorems are at the heart of *oriented matroids (OMs)*, which give a combinatorial model of linear algebra over ordered fields, it is natural to generalize Kirchberger's Theorem to them as well. We will prove it for *complexes of oriented matroids (COMs)*. COMs have been recently introduced in [1] as a common generalization of oriented matroids, affine oriented matroids, and lopsided sets. They can be simply described by a groundset E and a set of sign vectors $\mathcal{L}$. Even though the generalization from OMs to COMs omits some of the few axioms, it is still possible to prove the following version of Kirchberger's Theorem.

Kirchberger's Theorem for COMs: Let $\mathcal{M} = (E, \mathcal{L})$ be a COM of rank r and $|E| = n$. If for all $C \subseteq E$ with $|C| = r+1$ the sign vector $\{+\}^{|C|}$ is a sign vector of $\mathcal{M} \backslash (E \backslash C)$, then $\{+\}^{|C|}$ is a sign vector of $\mathcal{M}$.

This is the combinatorial version of Kirchberger's Theorem as we will explain in our lecture. Since the axioms that describe a COM are so simple, the proof is elementary and should be understandable for people who are unfamiliar with oriented matroids. It is an example of how new mathematical structures can be connected to findings from 120 years ago.

This is a joint work with Winfried Hochstättler (FernUniversität in Hagen) and Kolja Knauer (Universitat de Barcelona).

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Accurate Bidiagonal Decompositions of Structured Totally Nonnegative Matrices with Repeated Nodes

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The decomposition of a totally nonnegative matrix (one all of whose minors are nonnegative) as a product of nonnegative bidiagonals is a powerful tool for studying the properties of these matrices [1] as well as performing numerical computations to high relative accuracy [2].

The conventional bidiagonal decompositions resulting from the complete Neville elimination, when applied to totally nonnegative matrices of the Vandermonde or Cauchy type have singularities that are when some of the nodes defining those matrices coincide.

For example, the bidiagonal decomposition of a 3×3 Vandermonde matrix is:

$$\begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & \frac{z-y}{y-x} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y-x & \\ & & (z-x)(z-y) \end{bmatrix} \begin{bmatrix} 1 & x & \\ & 1 & y \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & x \\ & & 1 \end{bmatrix},$$

which is not defined when $x = y$.

By relaxing the requirement for the bidiagonal factors to have unit diagonals (which has no detrimental effects on our ability to study or compute with these matrices), the singularity at $x = y$ can be removed:

$$\begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & z-y \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & y-x & \\ & 1 & z-x \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & x & \\ & 1 & y \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & x \\ & & 1 \end{bmatrix}.$$

In this talk we will present the technique to systematically remove the singularities in the bidiagonal decompositions of many classes of structured totally nonnegative matrices with repeated nodes, such as (rational, h-, q-Bernstein-) Vandermonde, Lupas, Cauchy, Cauchy Vandermonde matrices, among others.

Practical examples of computing with these matrices will also be presented.

This is joint work with Jorge Delgado, Ana Marco, José-Javier Martínezd, Juan Manuel Peña, Per-Olof Persson, and Steven Spasov. Supported by the SJSU Woodward Fund.

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Perturbation theory of transfer function matrices

Lauri Nyman

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In [5], Tisseur defines a condition number for the eigenvalue of a polynomial matrix. To extend on this work, we define a structured condition number for a polynomial system matrix $P(\lambda)$ of rational matrices $R(\lambda)$. Since the zeros of rational matrices $R(\lambda)$ are the zeros of associated polynomial system matrices $P(\lambda)$ under minimality conditions [1, 4], this yields a way to characterize the sensitivity of a zero of $R(\lambda)$ to structured perturbations.

Any rational matrix $R(\lambda)$ can be written (or appears directly written) as a transfer function matrix. That is, of the form

$$R(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda), \tag{14}$$

where $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ are arbitrary polynomial matrices, with $A(\lambda)$ regular. Under minimality conditions [1, 4], the zeros of $R(\lambda)$ are the eigenvalues of the associated polynomial matrix

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix}. \tag{15}$$

The idea is to study the conditioning of zeros of rational matrices, allowing perturbations in the coefficients of the matrix polynomials in (14) such that their respective degrees are preserved. This yields a structured condition number for $P(\lambda)$ whose main difference with Tisseur's condition number is that the degrees of the block matrices are preserved separately. At least in some special cases [2], there are algorithms that guarantee that the backward error is structured precisely in this sense, and hence this structured condition number is relevant in practice.

When this structured condition number is compared with Tisseur's unstructured condition number for eigenvalues of matrix polynomials, it can be shown that the latter can be unboundedly larger. To capture all the zeros of $R(\lambda)$, regardless of whether they are poles or not, the notion of root vectors [3] can be considered.

This is ongoing joint work with Vanni Noferini (Aalto University), Javier Pérez (University of Montana) and María C. Quintana (Aalto University).

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Bi-additive Models: different types of distributions

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Motivated by classical cumulants and some properties, we explore models that are the sum of a fixed mean vector $X\underline{\beta}$ with w independent random terms $X_i\underline{Z}_i, i = 1, ..., w$. The random vectors $\underline{Z}_i, i = 1, ..., w$ will have $c_1, ..., c_w$ independent and identical distributed components, with variance $\sigma_1^2, ..., \sigma 2_w$. Thus the variance matrices of these models will be $\sum_{i=1}^{w} \sigma_i^2 M_i$, with $M_i = X_i X_i^\top$, $i = 1, ..., w$ and we will consider their first four cumulants. Its is often preferable to work with cumulants rather than moments, since the two are entirely equivalent and for independent random variables, the cumulants of a sum are the sums of the cumulants.

The types of the distributions of the component of vectors $\underline{Z}_i, i = 1, ..., w$ may be different, which makes the applications of these models not only centered on the normal type expanding its applications.

This is joint work with Sandra S. Ferreira (UBI), Dario Ferreira (UBI), and João T. Mexia (UNL).

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The Fiber of P-matrices: the Recursive Construction of All Matrices with Positive Principal Minors

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P-matrices have positive principal minors and include many well-known matrix classes (positive definite, totally positive, M-matrices etc.) How does one construct a generic P-matrix? Specifically, is there a characterization of P-matrices that lends itself to the tractable construction of every P-matrix? To answer these questions positively, a recursive method is employed that is based on a characterization of rank-one perturbations that preserve the class of P-matrices.

This is joint work with Faith Zhang

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Accurate computation of the inverse of Totally Positive collocation matrices of the Lupaş-type (p,q)-analogue of the Bernstein basis

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The collocation matrices of the Lupaş-type (p,q)-analogue of the Bernstein basis ((p,q)-Lupaş matrices in the sequel) are a generalization of the Vandermonde matrices obtained when replacing the monomial basis by a generalization of the Bernstein basis introduced in [1] and used in the area of CAGD: the Lupaş-type (p,q)-analogue of the Bernstein basis.

In this work we present a fast and accurate algorithm to compute the inverse of a strictly totally positive (p,q)-Lupaş matrix. Its first stage, which is the main contribution of this work, is the computation with high relative accuracy of the bidiagonal decomposition of the (p,q)-Lupaş matrix. Then, starting from this bidiagonal decomposition the inverse of the (p,q)-Lupaş matrix is also computed with high relative accuracy by using an algorithm developed by Marco and Martínez in [2].

The numerical experiments show the good properties of our approach, which gives very accurate results even when the condition number of the (p,q)-Lupaş matrices is very high.

This is joint work with Ana Marco (Universidad de Alcalá) and José-javier Martínez (Universidad de Alcalá). This research has been partially supported by Spanish Research Grant PGC2018-096321-B-I00 from the Spanish Ministerio de Ciencia, Innovación y Universidades. The authors are members of the Research Group ASYNACS *(Ref.CT-CE2019/683) of Universidad de Alcalá.*

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Weak majorization inequalities in Euclidean Jordan algebras

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In the setting of Euclidean Jordan algebra $\mathcal{V}$, we prove weak majorization inequalities

$$\lambda\big(|P_a(b)|\big) \underset{w}{\prec} \lambda\big(a^2\big) * \lambda\big(|b|\big) \quad \text{and} \quad \lambda\big(|a \circ b|\big) \underset{w}{\prec} \lambda\big(|a|\big) * \lambda\big(|b|\big),$$

for all a, $b \in \mathcal{V}$, where P_u and $\lambda(u)$ denote, respectively, the quadratic representation and the eigenvalue vector of u, and $\circ$ denotes the Jordan product in $\mathcal{V}$.

Extending these inequalities, given a linear map $T : \mathcal{V} \to \mathcal{V}$, we consider the set of all nonnegative vectors q in $\mathbb{R}^n$ with decreasing components that satisfy the pointwise weak majorization inequality

$$\lambda\big(|T(x)|\big) \underset{w}{\prec} q * \lambda\big(|x|\big).$$

With respect to the weak majorization ordering, we show the existence of the least vector in this set. Moreover, when T is a positive map, the least vector is shown to be the join (in the weak majorization order) of eigenvalue vectors of $T(e)$ and $T^*(e)$, where e is the unit element of the algebra.

In the form of applications, we prove the generalized Hölder type inequality

$$||a \circ b||_p \leq ||a||_r \, ||b||_s,$$

where p, q, $r \in [1, \infty]$ with $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$, and provide an estimate on the norm of a general linear map relative to spectral norms.

This talk is a brief summary of two papers in which the first is a joint work with Jiyuan Tao and M. Seetharama Gowda, and the second is a joint work with M. Seetharama Gowda. The presenting author is supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) No. 2021R1C1C2008350.

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Tropical Matrix Identities

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Tropical matrix theory is well known for its combinatorial nature. Applying the equivalency between graph theory and matrix theory, over the Max-Plus semiring, Butkovic showed [1] that the tropical setting gives interpretations to known combinatorial problems. We prove identities on compound matrices in extended tropical semirings. Such identities include analogues to properties of conjugate matrices, powers of matrices and Sylvester-Franke identity, all of which are of strong combinatorial flavor.

We then provide a new graph theoretic proof of the tropical Jacobi identity, connecting the compound of the inverse to the inverse of the compound. Following Butkovic's interpretations to tropical matrix identities, we develop an application of this theorem to optimal assignments with supervisions. That is, optimally assigning multiple tasks to one team, or daily tasks to multiple teams, where each team has a supervisor task or a supervised task.

This is joint work with S. Gaubert, M. Akian, S. Sergeev and M. MacCaig

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Geometric continuity, Riordan matrices and applications

Luis Felipe Prieto-Martínez

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Geometric continuity G^k is an elementary concept in the geometry of parametrized curves. It is a notion of smoothness that does not depend on a concrete parametrization, but on the curve itself.

A particular and important question related to this definition is the study of the smoothness of curves described by a piecewise C^k parametrization, that is, the smoothness on the union of smooth pieces. There are well known compatibility conditions (called the beta-constraints) for the parametrizations of each piece to guarantee such geometric continuity.

In this talk, we will explain how this compatibility conditions can be stated in terms of partial Riordan matrices (for $k < \infty$) and Riordan matrices (for $k = \infty$). Moreover, we will show how this new statement can help us to prove some uniqueness results concerning analytic curves. We will illustrate this method with two particular results related to well known problems in plane Geometry, namely, (1) if there exists an analytic curve with two interior equichordal points then it must be unique (related to the equichordal problem) and (2) the unique analytic curve with an exterior power point is the circle (related to Rosenbaum's Power Point Problem).

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Strong solvability of restricted interval systems and its applications in quadratic and geometric programming

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Interval systems of linear equations and inequalities are well studied and several concepts of solutions and solvability exist [1, 2, 3]. Our focus is mainly on strong solvability, which means solvability for each realization of interval values. In case of the interval inequalities, there is an interesting relation to the existence of a strong solution, defined analogously. Our aim is to extend the results to the situation, where we have an a priori restriction of the solutions to a given set D.

The motivation comes from the area of interval-valued optimization problems, where strong solvability means guaranteed feasibility of any realization of the problem. Strong solvability with strict inequalities then implies the robust Slater condition, which ensures that standard optimality conditions can be used.

We apply the issues particularly in two optimization classes, convex quadratic programming with quadratic constraints and posynomial geometric programming. Since the constraints are nonlinear here, we adapt the previous results by a suitable transformation and by utilizing the restricted domain D. For convex quadratic programming, we also utilize the presented result to improve a characterization of the worst case optimal value.

Eventually, we state several open problems that emerged while deriving the results; it is especially hard to involve interval linear equations in the characterization of strong solvability.

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$\star$-Lanczos procedure for non-autonomous ODEs

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The time ordered exponential $U(t)$ is the solution to the ODE

$$\frac{d}{dt}U(t) = A(t)U(t), \quad U(s) = I, \quad t \geq s,$$

where $A(t)$ is a time dependent matrix and s is the starting time. This ODE arises, e.g., from quantum dynamical systems, where $A(t, s)$ is related to the Hamiltonian of the system.
Computing the time ordered exponential remains a difficult problem and no satisfactory method is available for large-to-huge systems. Recently a new symbolic procedure based on walks on a graph was developed which expresses the solution as a continued fraction of finite depth and breadth [1]. This method is, however, prohibitively expensive for large problems.
The underlying graph can be approximated by using a symbolic Lanczos-like procedure, called $\star$-Lanczos [2], which results in a simpler graph with a tridiagonal adjacency matrix. At a lower computational cost an approximation to $U(t)$ can now be obtained via a Jacobi continued fraction. Underlying the $\star$-Lanczos procedure is a noncommutative algebra with a convolution-like product between functions, the $\star$-product. Variants of properties of the classical Lanczos iteration for matrices and vectors are valid for this Lanczos-like procedure for matrices of functions. For example, the moment matching property and the three term recurrence relation.
The $\star$-Lanczos procedure is a symbolic method and we aim to develop a numerical counterpart. Therefore, in this presentation, we discuss possible discretizations of this procedure. We deal with bivariate functions of the form $f(t)\Theta(t - s)$, with $\Theta(\cdot)$ the Heaviside step function. A suitable discretization expands such functions into a (double) series of orthogonal polynomials and transforms the $\star$-product into ordinary matrix-matrix multiplication. This allows us to use efficient numerical procedures to obtain an approximation to $U(t)$. One possible expansion basis for the series are the Legendre polynomials. However, due to the presence of a jump caused by the Heaviside function, we must deal with the Gibbs phenomenon.

This is joint work with Stefano Pozza (Charles University). Supported by Charles University Research program No. PRIMUS/21/SCI/009.

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Monge-like properties in the interval setting

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The Monge property of a real matrix $A \in \mathbb{R}^{n \times n}$ can be expressed as

$$a_{ij} + a_{k\ell} \leq a_{i\ell} + a_{kj}$$

for $1 \leq i \leq m$, $1 \leq j \leq n$. This simple to express property is fundamental for many efficient algorithms applied in various problems from geometry, combinatorics, optimisation or statistics [1, 2]. Many modifications of the property lead to interesting classes of Monge-like matrices, to name a few, *Robinsonian*, *ultrametric*, *totally monotonic*, or *Monge-permutable* matrices.

In our research, we do a systematic analysis of Monge-like properties in the interval setting. Matrix intervals allow for computations with inexact data. Rather than storing precise values (which might be impossible thanks to limitations of the measurement), we employ real intervals with a guarantee that each underlying value is in the range of its interval. Computations with matrix intervals are then carried out in a way that the exact solution of the original problem is guaranteed to be in the range of the output interval. Further, if the width of the interval is negligible, so is the error.

We focus on two variants of the Monge-like properties in the interval setting, so called *weak* and *strong* properties. For the definition of both of these properties, matrix realisations of the matrix interval (real matrices with entries from the intervals) are considered. If there is at least one realisation satisfying the Monge-like property, we say the matrix interval satisfies its weak form. If all of the realisations satisfy the Monge-like property, the matrix interval satisfies its strong form.

We deal with different characterisations of weak and strong Monge-like properties as well as with necessary and sufficient conditions. If the matrix interval satisfies the strong property, an interesting question to consider is if it can be recognised by checking the Monge-like property for a finite (hopefully, polynomial) number of its matrix realisations. This property of matrix intervals, referred to as the *interval property* [5], is also studied in our research. Finally, we investigate possible interval generalisations of known algorithm for Monge-like properties together with their complexity analysis.

This is joint work with prof. Jürgen Garloff (Konstanz). Supported by SVV 260578/2020.

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Average Mixing Matrices on Dutch Windmill Graphs

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Let A be the adjacency matrix of a graph. We may associate this graph with a continuous-time quantum walk by using a transition matrix $U(t) = \exp itA$. This allows us to create another matrix $\hat{M}$ which is independent of time and gives some measure of average probability values and long-term behavior. $\hat{M}$ is called the average mixing matrix, which we first saw in [1], but more work had been done prior by the same group in [2] and [3].

In our research, we've focused on cactus graphs, many of which are different from previous work done because they have repeated eigenvalues. We've shown what happens to the rank of $\hat{M}$ if we restrict our graphs to Dutch Windmill graphs, just one type of cactus graph. In this talk we will show that it has no better than half-rank and why, including the relationships between Dutch Windmills and path/star graphs.

This is joint work with Dr. Judi McDonald at Washington State University.

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Interval B-matrices, doubly B-matrices and B_{π}^{R}-matrices

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In 1968, mathematicians Cottle and Dantzig proposed the linear complementarity problem, denoted $LCP(M,q)$, where M is a matrix and q a vector. Later, Cottle et al. showed that for every vector q the $LCP(M,q)$ has a unique solution if and only if M is a P-matrix, i.e. all its principal minors are positive. However, verifying whether a given matrix is a P-matrix is co-NP-complete. Therefore several subclasses of P-matrices that are more easily recognizable are defined. Such classes might be B-matrices (introduced in [1]), doubly B-matrices (introduced in [2]) or B_{π}^{R}-matrices (introduced in [3]).

In our work, we generalize the three subclasses of P-matrices mentioned above into the interval setting. We define interval analogies of those classes and we deduce characterizations, both direct through some characteristic property or via reduction to finite number of real instances. That might help us again e.g. with the $LCP(M,q)$, this time with its interval variant, where we use intervals to somehow capture inaccuracy in data. That is because all the interval classes we derive are subclasses of interval P-matrices. What is interesting is that whereas the complexity of characterizations of interval B-matrices and interval B_{π}^{R}-matrices is the same as that of the real cases, which is $O(n^2)$, for interval doubly B-matrices it is $O(n^4)$ compared to $O(n^2)$ for the real case.

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Hub and authority centrality measures based on continuous-time quantum walks

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Measures of node centrality are a fundamental topic in network analysis. For directed networks, in particular, there is a distinction to be made between *hub* and *authority* centrality scores, since each node plays a double role in the network.

In recent years, interest for quantum computation has fueled the development of the theory of quantum walks on networks, which can be also used to define centrality measures on graphs. Building on ideas from [3] and [1], in this work we propose to employ continuous-time quantum walks (CTQW) to define measures of hub and authority centrality for directed graphs.

Recall that the time evolution of a CTQW on a graph is described by the Schrödinger equation

$$i\frac{\partial|\psi(t)\rangle}{\partial t} = H|\psi(t)\rangle, \tag{16}$$

where $|\psi(t)\rangle$ is the state of the system at time t, and the Hamiltonian operator H encodes the structure of the graph. The associated evolution operator takes the form $U(t) = \exp(-itH)$. Note that for quantum walks the quantum state of the system does not converge to a stationary state (as opposed to the classical case). For this reason, time averages are usually applied to define centrality scores.

We explore different choices for H and for the initial state $|\psi(0)\rangle$ and compare experimentally the resulting quantum centrality score with well-established centrality measures such as HITS, PageRank, the method from [1] and, when possible, discrete-time Quantum PageRank [4].

This is joint work with Roberto Grena (ENEA).

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Rational matrix solutions to $p(X) = A$

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Consider an $n \times n$ matrix A with rational entries, and let $p(\lambda)$ be a polynomial with rational coefficients. The question we consider is whether or not there is a rational $n \times n$ matrix X such that $p(X) = A$, and if there is, how to find it. In addition, we consider the problem over which field extension of the rationals there will be a solution, provided a solution over the complex numbers does exist.

The solution to this problem was first discussed in [1] for the case where A has n distinct eigenvalues. Our contribution is to extend this result to more general cases. This requires a new canonical form for the matrix A.

In the talk we will outline the steps involved in the solution of the problem, and if there is time, discuss some examples. In particular, we are interested in mth roots of A.

This is joint work with Gilbert Groenewald, Dawie Janse van Rensburg, Madelein van Straaten and Frieda Theron (all North-West University)

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Projections, L_p Norms and Stochastic Matrices for Ill-Conditioned Linear Systems of Equations

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In quite diverse application areas, we aim at finding a vector x, solution of a system of linear equations $Ax = b$, where b is a given vector and A is a given very ill-conditioned $m \times n$ matrix, making the resolution difficult. This issue appears, for example, in physics when discretizing a Fredholm integral equation of the first kind and in mathematical optimization when solving linear programs with interior point algorithms. One solving approach is to consider the system $RAx = Rb$, $Rb \in Im(RA)$, $Ker(RA) = Ker(A)$, where the preconditioning $n \times m$ matrix R is a gain matrix i.e. $\rho(I - RA\mid_{Im(I-RA)}) < 1$, with ρ the spectral radius of a matrix and I the identity matrix. Then, for any gain matrix R, the Richardson's iterative scheme $x^{k+1} = x^k + R(b - Ax),\ k = 1, 2, ...$ can be implemented to calculate a solution of the system. For any nonzero row matrix A, we can choose for R one of the two projective gain matrices: i) the Kaczmarz matrix $K = [K_1...K_n]$, with $K_i = \frac{1}{\|a_i\|_2^2}\prod_{j=1}^{i-1}(I - \frac{1}{\|a_j\|_2^2}a_j^* a_j)a_i^*$, where * is the transposed conjugate and the product is considered to be I whenever $i < 2$ and a_i is the ith row-vector of A; ii) the Cimmino matrix $C = \frac{2}{m}A^*D$, where D is a diagonal matrix with $D_{ii} = 1/\|a_i\|_2^2$. In addition, we propose the following approach for dealing with such an issue.

Firstly, using the L_1 norm, we construct an approximation R of a generalized inverse of a nonnegative matrix A such that the preconditioned matrix RA is stochastic ($RAe = e$, e the all-one vector). This property allows us to retrieve, in an original way, the Schultz-Hotelling-Bodewig's (SHB) algorithm (of order $q = 2$) of iterative refinement of the approximate inverse of a matrix: $R^0 = R$, $R^{k+1} = R^k\left(2I - AR^k\right),\ k = 1, 2, ...$ This basic approach is then extended to hermitian, semi-definite positive matrices and finally generalized to any complex rectangular matrices. The proposed preconditionning gain matrix R, has the general form $R = \alpha N_p^\nu A^* M_p^\mu$, where N_p, M_p are diagonal matrices involving a L_p norm related to A^* and A respectively and α, ν, μ are scalars (the Cimmino's matrix corresponds to the choice of the Euclidian norm in an asymmetrical structure : $\nu = 0;\ \mu = 2$ with $\alpha = \frac{2}{m}$). The proposed gain matrix with the norm L_1 and $\alpha = 1, \nu = \mu = 1$, always satisfies the convergence condition $\rho(I - RA\mid_{Im(I-RA)}) < 1$. Secondly, we propose a generalized iterative SHB scheme of any order $q \geq 2$, allowing to calculate, from any gain matrix R, successive approximations of the (generalized) inverse, denoted A^-, of $A \in \mathbb{C}_{\neq 0}^{m\times n}$, or $A \in \mathbb{C}_+^{m\times m}$, of rank m, based on the following theorem: $\lim\limits_{q\to\infty} R\sum_{j=1}^{q}\frac{q!}{j!(q-j)!}(-AR)^{j-1} = A^-$. By achieving q cycles of projections in Kaczmarz's and Cimmino's methods, this scheme accelerates convergence of row-action and Richardson schemes. The higher the order, the faster the convergence. But the gain in speed of convergence must be weighed against the very high cost for computing SHB matrices of order q and a compromise must be achieved. Regarding numerical results obtained on some pathological well-known test-cases (Hilbert, Nakasaka, ...), some of the proposed algorithms are empirically shown to be very efficient on ill-conditioned problems and robust to error propagation.

Bidiagonal decomposition and accurate computations with matrices of q-integers

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A matrix is totally positive if all its minors are nonnegative. A nonsingular totally positive matrix can be factorized as a product of nonnegative bidiagonal matrices. This factorization provides a natural parameterization of this class of matrices that can be used to perform many algebraic computations with high relative accuracy [3, 4], assuming that it can be computed with high relative accuracy. For example, it can be used to compute all the eigenvalues, all the singular values and the inverse to high relative accuracy. Quantum calculus is based on q-integers and has many applications (see [2]). In this talk, we will introduce some subclasses of totally positive matrices based on the q-integers like those of [1], for which the bidiagonal decomposition, and hence, the solution to the mentioned linear algebra problems, can be computed with high relative accuracy.

This is joint work with Juan Manuel Peña and Jorge Delgado (University of Zaragoza). Supported by the Spanish research Grant PGC2018-096321-B-I00 (MCIU /AEI)

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Solving (Overdetermined) Polynomial Equations

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Systems of polynomial equations arise in a wide range of (applied) mathematics and engineering applications, such as systems theory and control, numerical optimization, etc. Methods for solving systems of polynomial equations have been largely dominated by symbolic and hybrid symbolic-numeric approaches. Recent years have witnessed the (re)emergence of numerical solution methods [1, 2, 3] that are related to Sylvester and Macaulay matrices or resultants. In this framework, the system of polynomial equations can be viewed as a homogeneous linear matrix equation consisting of a large coefficient matrix that is multiplied with a vector of monomials. The solutions of the system of polynomial equations can be computed from a certain eigenvalue problem that is obtained from a numerical basis of the null space of the coefficient matrix.

The fact that this formulation expresses the problem of solving a system of polynomial equations in the language of (numerical) linear algebra suggests the exploration of finding approximate solutions of overdetermined systems of polynomial equations. The linear algebra approach that is described above is able to naturally deal with overdetermined systems of polynomial equations, provided that certain numerical rank decisions and projections are carefully considered throughout the solution procedure. The proposed method provides a fresh perspective on extending current methods for solving polynomial systems to the overdetermined case, which is a problem that received little research attention until now, likely because symbolic and hybrid-symbolic are not able to deal elegantly with overdetermined systems.

In this talk, we will develop the linear algebra-based solution method involving the Macaulay matrix formulation. Then we will illustrate how the method naturally extends to the case of overdetermined systems of polynomial equations.

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Hurwitz primitivity and synchronizing automata

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For each positive integer m, we use $[m]$ for the set of first m positive integers. Let $\mathcal{A} = (A_1, \dots, A_m)$ be an m-tuple of nonnegative $n \times n$ matrices. For each word α over $[m]$, say $\alpha = \alpha_1 \cdots \alpha_s$, we write $\mathcal{A}_\alpha$ for the product $A_{\alpha_1} \cdots A_{\alpha_s}$. We call $\mathcal{A}$ *primitive* if $\mathcal{A}_\alpha > 0$ for a nonempty word α over $[m]$. We call $\mathcal{A}$ *Hurwitz primitive* provided there exists a nonnegative integer vector $\tau = (\tau(1), \dots, \tau(m))$ such that for each $x, y \in [n]$ there exists a nonempty word $\alpha^{x,y}$ over $[m]$ such that $\mathcal{A}_{\alpha^{x,y}}(x, y) > 0$ and the number of occurrence of i in $\alpha^{x,y}$ is $\tau(i)$ for each $i \in [m]$. The m-tuple τ satisfying the above property is named a *Hurwitz primitive vector* of $\mathcal{A}$.

Let NZ_1 denote the set of nonnegative matrices without zero rows and let NZ_2 denote the set of nonnegative matrices without zero rows/columns. We give a unified combinatorial proof for the Protasov-Vonyov characterization [5] of primitive NZ_2-matrix tuples and the Protasov characterization [3] of Hurwitz primitive NZ_1-matrix tuples. By establishing a connection with synchronizing automata, for any Hurwitz primitive m-tuple $\mathcal{A}$ of $n \times n$ NZ_1-matrices we give an $O(n^3m^2)$-time algorithm to find a Hurwitz primitive vector τ of $\mathcal{A}$ such that $\sum_{i \in [m]} \tau(i) = O(n^3)$. For any given m-tuple of $n \times n$ NZ_2-matrices, we present an $O(n^2m)$-time algorithm to test whether or not it is primitive.

This is joint work with Yaokun Wu (Shanghai Jiao Tong University).

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Efficient Computation of Parametric Reduced Order Models using Randomization

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Partial differential equations-based nonlinear parametric inverse problems appear in many applications. The main computational bottleneck in these problems is the repeated evaluation of the large-scale forward model, which often requires solving large linear systems for many source terms as well as multiple frequencies and wavelengths at each optimization step. In addition, for Newton-type methods, which may be required for fast convergence, the solution of additional linear systems with the adjoint operator may be required to efficiently compute derivative information. As rapid advances in technology allow for large numbers of sources and detectors, these problems become computationally prohibitively expensive.

We have successfully used reduced order models (ROM) to drastically reduce the size of the linear systems while still obtaining accurate solutions. However, even the construction of the ROM bases incurs a substantial cost, as it requires the solution of large linear systems for all sources, frequencies, and detectors for interpolation points in parameter space to build a candidate basis for the ROM projection space. We propose to use randomization to approximate this low-rank candidate basis efficiently and drastically reduce the number of large linear solves for constructing the global ROM basis. We also analyze the low-rank structure of the candidate basis for our problem of interest, diffuse optical tomography. The ideas presented are relevant to many other large scale inverse problems and optimization problems.

This is joint work with Selin Aslan (Argonne National Lab and Virginia Tech) and Serkan Gugercin (Virginia Tech). This work was supported by the National Science Foundation under Grants DMS-1720305 and DMS-1438768 and by the Simons Foundation Grant 507536.

The change of the Weierstrass structure under one row perturbation

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We study the change of the structure of a regular pencil when we perform small perturbations of some of its rows, while the rest of the rows remain unaltered. We provide necessary conditions when several rows are perturbed, and prove them to be sufficient to prescribe the homogenous invariant factors or the Weyr characteristic of the resulting pencil when one row is perturbed.

We generalize to regular pencils previous studies in the field. Changes in the similarity invariants of a matrix when small additive perturbations are performed over one or several rows have been analyzed in [1, 2], and changes in the feedback invariants of a pair of matrices have also been explored in [3]).

This is joint work with Itziar Baragaña (Universidad del País Vasco / Euskal Herriko Unibertsitatea, UPV/EHU).

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Construction of a sequence of orthogonal rational functions

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Orthogonal polynomials are an important tool to approximate functions. Orthogonal rational functions provide a powerful alternative if the function of interest is not well approximated by polynomials. Polynomials orthogonal with respect to certain discrete inner products can be constructed by applying the Lanczos or Arnoldi iteration to appropriately chosen diagonal matrix and vector. This can be viewed as a matrix version of the Stieltjes procedure. The generated nested orthonormal basis can be interpreted as a sequence of orthogonal polynomials. The corresponding Hessenberg matrix, containing the recurrence coefficients, also represents the sequence of orthogonal polynomials.

Alternatively, this Hessenberg matrix can be generated by an updating procedure. The goal of this procedure is to enforce Hessenberg structure onto a matrix which shares its eigenvalues with the given diagonal matrix and the first entries of its eigenvectors must correspond to the elements of the given vector. Plane rotations are used to introduce the elements of the given vector one by one and to enforce Hessenberg structure.

The updating procedure is stable thanks to the use of unitary similarity transformations. In this talk rational generalizations of the Lanczos and Arnoldi iterations are discussed. These iterations generate nested orthonormal bases which can be interpreted as a sequence of orthogonal rational functions with prescribed poles. A matrix pencil of Hessenberg structure underlies these iterations. We show that this Hessenberg pencil can also be used to represent the orthogonal rational function sequence and we propose an updating procedure for this case. The proposed procedure applies unitary similarity transformations and its numerical stability is illustrated.

This is joint work with Niel Van Buggenhout and Marc Van Barel.

H-selfadjoint mth roots of H-selfadjoint matrices over the quaternions

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Consider a square matrix B in the indefinite inner product space generated by an invertible Hermitian matrix H. The matrix B is called H-selfadjoint if it is selfadjoint in the corresponding indefinite inner product space, or equivalently, if $HB = B^*H$.

Let B be an H-selfadjoint complex matrix. We give the necessary and sufficient conditions for the existence of an H-selfadjoint matrix A such that $A^m = B$, that is, A is an mth root of B.

We will look at the cases where the indefinite inner product is defined on a complex vector space and where it is defined on a quaternion vector space.

This is joint work with A.C.M. Ran (VU Amsterdam and North-West University), G.J. Groenewald, D.B. Janse van Rensburg, and F. Theron (all North-West University).

An alternative canonical form for quaternionic H-unitary matrices.

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The field of linear algebra over the quaternions is a research area which is still in development. In this paper we continue our research on canonical forms for a matrix pair (A, H), where the matrix A is H-unitary, H is invertible and with A as well as H quaternionic matrices. We seek an invertible matrix S such that the transformations from (A, H) to $(S^{-1}AS, S^{*}HS)$ brings the matrix A in Jordan form and simultaneously brings H into a canonical form. Canonical forms for such pairs of matrices already exist in the literature, the goal of the present paper is to add one more canonical form which specifically keeps A in Jordan form, in contrast to the existing canonical forms.

This is joint work with G.J. Groenewald (North-West University, SA), A.C.M. Ran (VU, the Netherlands). Supported by the DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences, ref nr. 2022-012-ALG-ILAS.

The combinatory under isomorphic lattices of hyperinvariant subspaces

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Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a linear transformation over the complex field and $Hinv(f)$ the lattice of the hyperinvariant subspaces of f (that is, the set of linear transformations commuting with f). We study the linear transformations whose lattices of hyperinvariant subspaces are isomorphic to $Hinv(f)$. We present a revision of the results provided in [1].

This is joint work with David Mingueza (Nestlé) and Alicia Roca (Universidad Politécnica de Valencia). Supported by the Spanish MICINN research project PID2019-104047GB-I00.

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About the type of broom trees

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Fiedler ([1]) provides a classification of trees according to whether there is an eigenvector corresponding to the algebraic connectivity which has a zero entry by using the concept of characteristic vertices. Grone and Merris ([2]) denoted those two classes of trees as type 1 and type 2. Another approach to trees and their classification is given by Kirkland, Neumann and Shader in [5].

For trees, to identify families for which its elements are of the same type is not an easy task. We consider the broom tree $T_{n,k}$ of order n, obtained by the coalescense of one leaf of the path of order $n-k$ with the center of the star of order $k+1$. In [3] Patra shows conditions for a broom tree by of type 2, proving that $T_{n,2}$ is of type 2 and giving a lower bound for k depending on n in order to guarantee that $T_{n,k}$ the 2 type. In [4] is conjectured that broom trees are of type 2.

In this work some theorical and experimental results for the type of some subfamilies of broom trees are presented. The results are based on the characterization given by [5].

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Weighted Projections of Alternating Sign Matrices and Latin-like Squares

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To any $n \times n$ Latin square L, we may associate a sequence of $n \times n$ permutation matrices $P = P_1, \ldots, P_n$ such that

$$L = L(P) = \sum_{k=1}^{n} kP_k.$$

Brualdi and Dahl [1] introduced a generalisation of a Latin square, called an *alternating sign hypermatrix Latin-like square (ASHL)*, obtained by replacing P in the above weighted sum with an *alternating sign hypermatrix (ASHM)*. An ASHM is an $n \times n \times n$ hypermatrix with entries from $\{1, -1, 0\}$ such that the non-zero entries in each row, column, and vertical line alternate in sign, beginning and ending with $+1$.

Alternating sign matrices arise in a number of different contexts as a natural generalisation of permutation matrices, and every sequence of $n \times n$ permutation matrices corresponding to a Latin square forms the planes of a unique $n \times n \times n$ ASHM. This generalisation therefore follows very naturally from the above interpretation of a Latin square, with an ASHM A has corresponding ASHL L defined as follows.

$$L = L(A) = \sum_{k=1}^{n} kA_k,$$

where A_k is the k^{th} plane of A.

As a step towards characterising these Latin-like squares without needing to find the underlying hypermatrix, we can consider the *weighted projection* [1] of an alternating sign matrix. The weighted projection sends an ASM A to a vector $v(A)$, which corresponds to a single row or column of a Latin-like square.

This talk presents proof of a conjecture [1] that for any vector v which is *majorized* by $(n, n-1, \ldots, 3, 2, 1)$, there exists an alternating sign matrix A for which $v(A) = v$, and discusses further steps towards characterising ASHLs [2].

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Poster Sessions

Organiser: Ronan Egan

23 June	10:00	**Blake McGrane-Corrigan**	p297
Diffusive Stability, Common Lyapunov Functions and Leslie Matrices			
23 June	10:00	**John Stewart Fabila-Carrasco**	p294
The Cartesian product of graphs and entropy metrics for graph signals.			
23 June	10:00	**Paula Kimmerling**	p296
Recursion of Eigenvectors in Dutch Windmill Graphs			
23 June	10:00	**Priyanka Joshi**	p295
Powers of Karpelevič Arcs			
23 June	10:00	**V A Kandappan**	p299
Hierarchical Off Diagonal Low Rank Matrices (HODLR) for problems in higher dimensions			
23 June	10:00	**Victoria Sánchez Muñoz**	p298
The Mathematics behind the quantification of entanglement in Quantum Mechanics			

The Cartesian product of graphs and entropy metrics for graph signals.

John Stewart Fabila-Carrasco

University of Edinburgh

Entropy metrics are nonlinear measures to quantify the complexity of time series. Among them, Permutation Entropy (PE) is a well-established nonlinear metric based on the comparison of neighbouring values within patterns in a time series. PE is robustness to noise and fast computation [1]. Multivariate entropy metrics techniques are needed to analyse data consisting of more than one time series. To this end, we present a multivariate permutation entropy, $\mathrm{MPE}_{\mathrm{G}}$, using a graph-based approach.

Given a multivariate signal, the algorithm MPE_G introduced in [2] involves two main steps:

1) Graph construction: we construct an underlying graph G as the Cartesian product of two graphs G_1 and G_2, i.e., $G := G_1 \square G_2$, where G_1 preserves temporal information of each times series together with G_2 that models the relations between different channels.

2) Permutation entropy for graph signals: we consider the multivariate signal as samples defined on the regular graph G and apply the recently introduced permutation entropy for graphs PE_{G} [3].

PE_{G} is an entropy metric to analyse signals measured over irregular graphs by generalising permutation entropy. The algorithm PE_{G} is based on comparing signal values on neighbouring vertices, using the adjacency matrix, and it has important relations with the sign of the Laplacian matrix [4]. This generalisation preserves the properties of classical permutation for time series and the recent permutation entropy for images, and it can be applied to any graph structure with synthetic and real signals.

Our graph-based approach to multivariate permutation entropy gives the flexibility to consider diverse types of cross channel relationships and signals, and it overcomes with the limitations of current multivariate permutation entropy algorithms.

This is joint work with Javier Escudero (University of Edinburgh) and Chao Tan (Tianjin University). Supported by the Leverhulme Trust via a Research Project Grant (RPG-2020-158), and Post-Doctoral Enrichment Award from the Alan Turing Institute to JSFC.

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Powers of Karpelevič Arcs

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A celebrated result of Karpelevič describes Θ_n the collection of all eigenvalues arising from the stochastic matrices of order n. The boundary of Θ_n is a disjoint union of arcs, known as the Karpelevič arcs.

Johnson and Paparella [2] considered relationships between different arcs, and posed a conjecture on their powers. This conjecture was later proved by Kim and Kim [3]. We continue their work and give a complete characterization of the Karpelevič arcs that are powers of some other Karpelevič arc. Furthermore, we study the powers of the corresponding realising matrices. In particular, we show that in the case when a Karpelevič arc is a power of another Karpelevič arc, only selected corresponding realising matrices can be written as a power of another stochastic matrix.

This is joint work with Stephen Kirkland (University of Manitoba) and Helena Šmigoc (University College Dublin). Supported by Science Foundation Ireland (SFI) under Grant Number SFI 18/CRT/6049.

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Recursion of Eigenvectors in Dutch Windmill Graphs

Paula Kimmerling

Washington State University

Let A be the adjacency matrix of a graph. We may associate this graph with a continuous-time quantum walk by using a transition matrix $U(t) = \exp itA$. This allows us to create another matrix $\hat{M}$ which is independent of time and gives some measure of average probability values and long-term behavior. $\hat{M}$ is called the average mixing matrix, which we first saw in [1], but more work had been done prior by the same group in [2] and [3].

In our research, we've focused on cactus graphs, many of which are different from previous work done because they have repeated eigenvalues. We've shown what happens to the rank of $\hat{M}$ if we restrict our graphs to Dutch Windmill graphs, just one type of cactus graph. In this poster we will discuss one of our proof techniques to supplement our main result in our talk, which is that the rank is no better than half. This involves a recursive relationship between some of the entries of the eigenvectors.

This is joint work with Dr. Judi McDonald at Washington State University.

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Diffusive Stability, Common Lyapunov Functions and Leslie Matrices

Blake McGrane-Corrigan

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How does connecting two stable linear time-invariant systems affect the stability of the resulting coupled system? This question can arise in ecological applications, for example when investigating the effects of dispersal/diffusion in patchy environments. Inspired by recent work on establishing conditions for robust diffusive stability via common linear copositive Lyapunov functions, we present some results relating robust diffusive stability to other types of Lyapunov functions. We further show that when any pair of linear systems described by Leslie matrices are diffusively coupled, they are stable for any choice of coupling matrix.

This is joint work with Oliver Mason and Rafael de Andrade Moral (Maynooth University) and is supported by the Irish Research Council through a Government of Ireland Postgraduate Scholarship.

The Mathematics behind the quantification of entanglement in Quantum Mechanics

Victoria Sánchez Muñoz

National University of Ireland Galway

The entanglement of a quantum state is one of the unique features of Quantum Mechanics in comparison to classical physics. Many of the novel phenomena and applications that are found in Quantum Cryptography, Quantum Information, and Quantum Computing arise from the concept of entanglement (see the comprehensive review [1]). That is why the classification and quantification of entanglement is of high importance in Quantum Mechanics.

The present poster illustrates how entanglement is defined, how it is classified, and the Mathematics behind its quantification for a two-qubit and three-qubit pure quantum states.

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Hierarchical Off Diagonal Low Rank Matrices (HODLR) for problems in higher dimensions

V A Kandappan

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Hierarchical matrices such as HODLR, HSS, $\mathcal{H}^2$, etc. are used in constructing approximations and computing matrix operations for rank structured matrices. We introduced a new class of Hierarchical matrices [1] for matrices arising out of the discretization of PDEs in 2D, in this work we extend this to problems from higher dimensions and applications in higher dimensional statistics, machine learning, etc. The matrix partitioning in a d-dimensional setting is done by constructing a 2^d-tree over the underlying computational domain. We present the growth of ranks for various kinds of interactions. We identify the sub-blocks whose ranks do not scale with their size and low-rank approximate them. As a result, the computational complexity of construction and matrix-vector product scale almost linearly in the system size. We present various benchmarks and compare its performance with other Hierarchical matrices.

This is joint work with Vaishnavi Gujjula (Indian Institute of Technology Madras) and Sivaram Ambikasaran (Indian Institute of Technology Madras)

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