AARMS-CRM Workshop on NA of SPDEs, July 2016

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Short course on Numerical Analysis of Singularly Perturbed Differential Equations

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§3 Coupled systems of SPDEs

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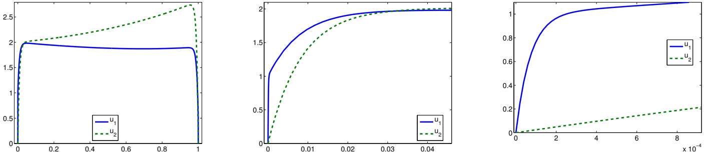


Handout version.

Outline

	Monday, 25 July	Tuesday, 26 July
09:00	Welcome/Coffee	
09:15	1. Introduction to singularly perturbed problems	5. PDEs (i): time-dependent problems.
10:00	Break	
10:15	2. Numerical methods and uniform convergence; FDMs and their analysis.	6. PDEs (ii): elliptic problems 7. Finite Element Methods
12:00	*Lunch*	
14:00	3. Coupled systems	8. Convection-diffusion (Stynes)
15:00	Break	
15:15	3. Coupled systems (continued)	9. Nonlinear problems (Kopteva)
16:15	4. Lab 1	10. Lab 2 (PDEs)
17:30	*Finish*	

(≈ 2 hours.)

In this section we'll consider coupled systems of two equations, with interacting boundary layers.

We'll see how to derive solution decompositions and how to use these to construct suitable meshes.

We'll then move onto coupled systems of ℓ-equations: techniques for generalisation.

To finish, we'll look at how to construct both piecewise uniform and graded meshes for these problems.

Primary references

Some of the content of this presentation is based on...

- [Madden and Stynes, 2003], for bounds on derivatives of the true solution to a coupled system of two equations.
- [Kellogg et al., 2008], for ideas on extension to larger systems.
- [Linß and Madden, 2009], for graded meshes.

For a more detailed exposition see [Linß and Stynes, 2009] and, especially, [Linß, 2010].

The study of numerical methods for singularly perturbed systems dates back to the pioneering work of [Bakhvalov, 1969]. See also [Shishkin, 1995].

Coupled systems

The following formulation of a general system of $\ell \geqslant 2$ singularly perturbed problems is presented by [Linß and Stynes, 2009] as

$$-\mathrm{diag}(\boldsymbol{\varepsilon})\Delta\mathbf{u} - \mathbf{A}\cdot\nabla\mathbf{u} + B\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = g,$$

where

- $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_\ell)^T$ is a set of perturbation parameters,
- $\mathbf{A} = (A_1, A_2)$, and A_1, A_2 and B are matrix-valued functions.
- $\mathbf{f}$ and $\mathbf{g}$ are vector-valued functions.

Coupled systems

$$-\mathrm{diag}(\varepsilon)\Delta\mathbf{u} - \mathbf{A}\cdot\nabla\mathbf{u} + B\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = g.$$

Classification [Linß and Stynes, 2009]

(i) Reaction-diffusion: $-\mathrm{diag}(\varepsilon)\Delta\mathbf{u} + B\mathbf{u} = \mathbf{f}$.

(ii) Weakly coupled convection-reaction-diffusion:
$-\mathrm{diag}(\varepsilon)\Delta\mathbf{u} + \mathrm{diag}(\mathbf{a})\cdot\nabla u + B\mathbf{u} = \mathbf{f}$.

(iii) Strongly coupled convection-reaction-diffusion:
$-\mathrm{diag}(\varepsilon)\Delta\mathbf{u} + \mathbf{A}\cdot\nabla u + B\mathbf{u} = \mathbf{f}$.

"Each subclass has its own peculiarities".

We will focus on the simplest setting: reaction-diffusion problems with $\Omega = (0, 1)$. Also, we'll begin with $\ell = 2$.

Example (A coupled system of reaction-diffusion equations)

$$-\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}^2 \mathbf{u}'' + B(x)\mathbf{u} = \mathbf{f} \text{ on } (0,1), \quad \text{with } \mathbf{u}(0) = \mathbf{u}(1) = 0.$$

In spite of its simplicity, there is much that can be learned from this problem, which itself is often reduced to three sub-classes:

(a) $\varepsilon_1 = \varepsilon_2 \ll 1$ (The single parameter problem)

(b) $\varepsilon_1 \ll \varepsilon_2 = 1$ (One small parameter)

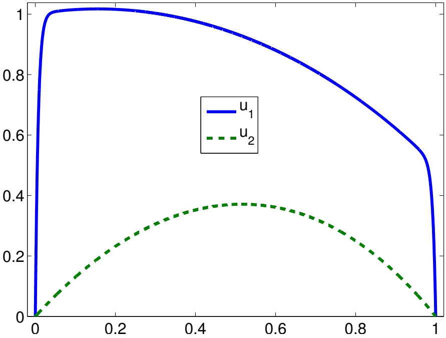
(c) $\varepsilon_1 \ll \varepsilon_2 \ll 1$ (Two small parameters)

Case (a), i.e., the *single parameter problem*, is the least interesting. Under reasonable assumptions on B, most techniques (numerical and mathematical) for uncoupled problems extend directly to this case.

Nonetheless, the single parameter problem can be a good starting point, particularly for larger systems.

Example (Case (b): $\varepsilon_1 \ll \varepsilon_2 = 1$)

$$-\begin{pmatrix} 10^{-2} & 0 \\ 0 & 1 \end{pmatrix}^2 \mathbf{u}'' + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 2-x \\ 1+e^x \end{pmatrix} \text{ on } (0,1), \quad \text{with } \mathbf{u}(0) = \mathbf{u}(1) = 0.$$

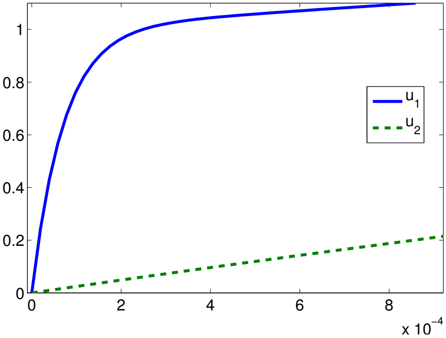


- The component u_1 features (strong) layers, of width $\mathcal{O}(\varepsilon)$.
- u_2 features "weak" layers: u_2' and u_2'' are bounded independent of ε, but u_2''' is not.

This is the most interesting case, since solutions possess multiple, interacting layers.

Example (Case (b): $\varepsilon_1 \ll \varepsilon_2 \ll 1$)

$$-\begin{pmatrix} 10^{-4} & 0 \\ 0 & 10^{-2} \end{pmatrix}^2 \mathbf{u}'' + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 2-x \\ 1+e^x \end{pmatrix} \text{ on } (0,1), \quad \mathbf{u}(0) = \mathbf{u}(1) = 0.$$



- Both components clearly features layers of width $\mathcal{O}(\varepsilon_2)$.
- u_1 also features a layer of width $\mathcal{O}(\varepsilon_1)$.
- Much of the mathematical interest/difficulty comes from the multi-scale behaviour.

A system of two equations

The model problem (again)

$$\mathcal{L}\mathbf{u} := \begin{pmatrix} -\varepsilon^2 \frac{d^2}{dx^2} & 0 \\ 0 & -\mu^2 \frac{d^2}{dx^2} \end{pmatrix} \mathbf{u} + B\mathbf{u} = \mathbf{f}, \qquad \text{on } \Omega = (0,1),$$

where

$$B = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}, \quad \mathbf{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},$$

We shall assume, for now, that, for all $x \in \Omega$,

$$b_{ik}(x) \begin{cases} > 0 & i = j \\ \leqslant 0 & i \neq j \end{cases} \qquad \text{and} \qquad \sum_j b_{ij} \geqslant \beta^2 > 0,$$

for some $\beta > 0$.

We shall see that

- The numerical scheme for the uncoupled problem generalises in the obvious way;
- The numerical analysis takes *much* more care.

The FDM and layer-adapted (Shishkin) mesh

Mesh: $0 = x_0 < x_1 < \cdots < x_N = 1$

Step sizes: $h_i = x_i - x_{i-1}$

Discretization:

$$L^N \mathbf{U} := \begin{cases} -\varepsilon^2 \left(\delta^2 U_1\right)_i + b_{11,1} U_{1,i} + b_{12,i} U_{2,i} & = f_{1,i} \\ -\mu^2 \left(\delta^2 U_2\right)_i + b_{21,1} U_{1,i} + b_{22,i} U_{2,i} & = f_{2,i} \end{cases}$$

$$U_{1,0} = U_{1,N} = U_{2,0} = U_{2,N} = 0.$$

where

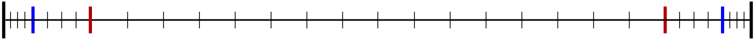
$$\left(\delta^2 v\right)_i = \frac{2}{h_{i+1} + h_i} \left(\frac{v_{i+1} - v_i}{h_{i+1}} - \frac{v_i - v_{i-1}}{h_i} \right)$$

The FDM and layer-adapted (Shishkin) mesh

Mesh Transition Points, always assuming $\varepsilon \leqslant \mu$:

$$\tau_\mu = \min\left\{\frac{1}{4}, 2\frac{\mu}{\beta}\ \ln N\right\},$$

$$\tau_\varepsilon = \min\left\{\frac{1}{8}, \frac{\tau_\mu}{2}, 2\frac{\varepsilon}{\beta}\ \ln N\right\}.$$



The FDM and layer-adapted (Shishkin) mesh

Numerical results demonstrate that the scheme is robust.

ε	128	256	512	1024	2048
10^{-2}	5.69e-02	1.96e-02	6.29e-03	1.96e-03	5.94e-04
10^{-3}	6.17e-02	1.95e-02	6.28e-03	1.96e-03	5.94e-04
10^{-4}	8.88e-02	3.28e-02	1.14e-02	3.71e-03	1.16e-03
10^{-5}	8.88e-02	3.28e-02	1.14e-02	3.71e-03	1.16e-03
10^{-6}	8.88e-02	3.28e-02	1.14e-02	3.71e-03	1.16e-03
10^{-7}	9.01e-02	3.13e-02	1.01e-02	3.10e-03	9.24e-04
10^{-8}	9.02e-02	3.13e-02	1.01e-02	3.10e-03	9.24e-04
10^{-9}	9.02e-02	3.13e-02	1.01e-02	3.10e-03	9.24e-04
10^{-10}	9.02e-02	3.13e-02	1.01e-02	3.10e-03	9.24e-04
E^N	9.02e-02	3.13e-02	1.01e-02	3.10e-03	9.24e-04
p	1.46	1.52	1.62	1.68	1.72
C	62.8	69.9	76.8	81.0	83.6

Analysis

That's all very well in practice... but how does it work in theory?

We'll outline techniques to proving that the method is robust. Again this depends on

- A solution decomposition, and sharp pointwise bounds on components of the decomposition.
- *Maximum Principles+Barrier Functions*.

Bored?

1. Show that the operators $\mathcal{L}$ and L^N satisfy maximum principles.
2. Can you generalise your proof with weaker assumptions on B?
3. Can you generalise your proof to systems on ℓ equations?

Construct problems with solutions that represent the "smooth" and "layer-parts" of the solution separately:

Let

$$\mathbf{u} = \mathbf{v} + \mathbf{w},$$

where $\mathbf{v}$ is the solution to the problem

$$\mathcal{L}\mathbf{v} = \mathbf{f} \text{ on } \Omega, \qquad \mathbf{v} = B^{-1}\mathbf{f} \text{ on } \partial\Omega \quad \textit{(smooth part)},$$

and w is the solution to the problem

$$\mathcal{L}\mathbf{w} = \mathbf{0} \text{ on } \Omega, \qquad \mathbf{w} = \mathbf{u} - \mathbf{v} \text{ on } \partial\Omega \quad \textit{(singular part)}.$$

Then it can be shown that, for $k = 0, 1, 2, 3$,

$$\|\mathbf{v}_1^{(k)}\| \leqslant C(1 + \varepsilon^{(2-k)}) \quad \text{and} \quad \|\mathbf{v}_2^{(k)}\| \leqslant C(1 + \mu^{(2-k)}).$$

And if we define

$$\mathcal{B}_{\varepsilon}(x) = \exp(-x\alpha/\varepsilon) + \exp((x-1)\alpha/\varepsilon),$$

$$\mathcal{B}_{\mu}(x) = \exp(-x\alpha/\mu) + \exp((x-1)\alpha/\mu).$$

Then, for example,

$$|w_1(x)| \leqslant C\mathcal{B}_{\mu}(x), \qquad |w_2(x)| \leqslant C\mathcal{B}_{\mu}(x),$$
$$|w_1'(x)| \leqslant C\big(\varepsilon^{-1}\mathcal{B}_{\varepsilon}(x) + \mu^{-1}\mathcal{B}_{\mu}(x)\big), \qquad |w_2'(x)| \leqslant C\mu^{-1}\mathcal{B}_{\mu}(x)$$
$$|w_1''(x)| \leqslant C\big(\varepsilon^{-2}\mathcal{B}_{\varepsilon}(x) + \mu^{-2}\mathcal{B}_{\mu}(x)\big), \qquad |w_2''(x)| \leqslant C\mu^{-2}\mathcal{B}_{\mu}(x)$$
$$|w_1'''(x)| \leqslant C\big(\varepsilon^{-3}\mathcal{B}_{\varepsilon}(x) + \mu^{-3}\mathcal{B}_{\mu}(x)\big), \qquad |w_2'''(x)| \leqslant C\big(\varepsilon^{-1}\mu^{-2}\mathcal{B}_{\varepsilon}(x) + \mu^{-3}\mathcal{B}_{\mu}(x)\big).$$

This last term is the most complicated/tedious to deal with, since it mixes terms with two scales: **it requires a further decomposition**.

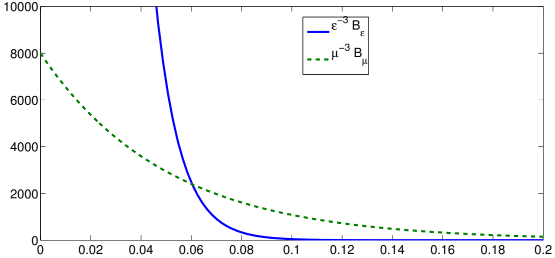
If $\varepsilon = \mu$, nothing further is needed. So take $\varepsilon < \mu$. The decompose $\mathbf{w}$ as

$$w_1(x) = w_{1,\varepsilon}(x) + w_{1,\mu}(x), \qquad w_2(x) = w_{2,\varepsilon}(x) + w_{2,\mu}(x),$$

where

$$|w''_{1,\varepsilon}(x)| \leqslant C\varepsilon^{-2}\mathcal{B}_\varepsilon(x), \qquad |w''_{2,\varepsilon}(x)| \leqslant C\mu^{-2}\mathcal{B}_\varepsilon(x),$$
$$|w'''_{1,\mu}(x)| \leqslant C\mu^{-3}\mathcal{B}_\mu(x), \qquad |w'''_{2,\mu}(x)| \leqslant C\mu^{-3}\mathcal{B}_\mu(x).$$

"Proof": this construction then relies on several ideas, including that there is a point $x^\star \in (0, 1/2)$ where $\varepsilon^{-3}\mathcal{B}_\varepsilon(x^\star) = \mu^{-3}\mathcal{B}_\mu(x^\star)$.



The remaining key ideas for the analysis include:

- Constructing barrier functions piecewise on $(0, x^\star)$, $(x^\star, 1 - x^\star)$, $(1 - x^\star)$ to complete the decomposition.
- Because derivatives of the "regular" component, $\mathbf{v}$, do not depend badly on ε and μ we can apply standard analysis.
- In the regions $[0, \tau_\varepsilon]$ and $[\tau_\varepsilon, \tau_\mu]$, the mesh is sufficiently fine to analyse the appropriately decomposed solution of the "layer" component, $\mathbf{w}$.
- In $[\tau_\mu, 1 - \tau_\mu]$, the layer part has decayed so that $|\mathbf{w}| \leqslant CN^{-2}$.

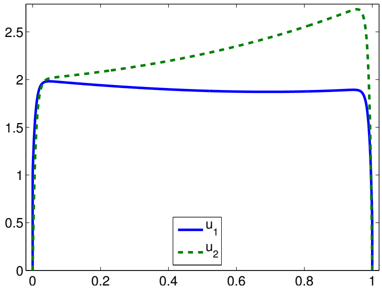
Extension to larger systems

We will finish by considering how the methods and analysis can be extended to larger systems of $\ell \geqslant 2$ equations, and focus on two issues

1. establishing stability of the continuous and discrete operator;
2. How to construct a mesh when the solution has numerous overlapping layers.

Example (Recall our example from the start of this section)

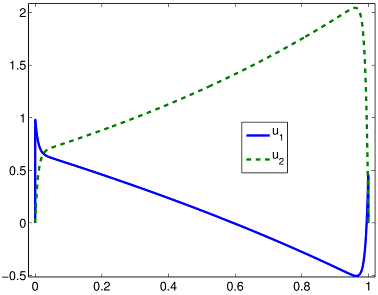
$$-(10^{-4})^2 u_1'' + 2u_1 - u_2 = 2 - x \qquad u_1(0) = u_1(1) = 0$$
$$-(10^{-2})^2 u_2'' - u_1 + 2u_2 = 1 + e^x \qquad u_2(0) = u_2(1) = 0$$



Now let's consider what happens when we violate the assumption that b_{12} and b_{21} are non-positive.

Example (Now let's change the reaction coefficients)

$$-(10^{-4})^2u_1'' + 2u_1 + u_2 = 2 - x \qquad u_1(0) = u_1(1) = 0$$
$$-(10^{-2})^2u_2'' + u_1 + 2u_2 = 1 + e^x \qquad u_2(0) = u_2(1) = 0$$



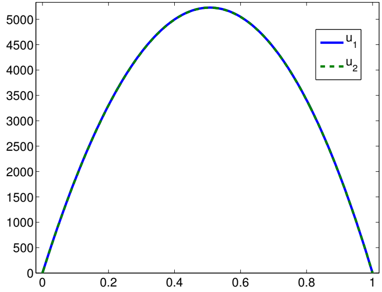
Even though the right-hand sides of the equations are positive, we obtain negative solutions. So the associated differential operator does *not* satisfy a maximum principle.

However, the solution remains stable.

Next, let's violate the assumption that $b_{11} + b_{12}$ and $b_{21} + b_{22}$ bounded away from zero.

Example (Now let's change the reaction coefficients)

$$-(10^{-4})^2 u_1'' + u_1 - u_2 = 2 - x \qquad u_1(0) = u_1(1) = 0$$
$$-(10^{-2})^2 u_2'' - u_1 + u_2 = 1 + e^x \qquad u_2(0) = u_2(1) = 0$$



It appears that this operator is *not* (ε, μ)-stable.

To examine this further, we return to the general problem

$$\mathcal{L}\mathbf{u} := -\mathrm{diag}(\boldsymbol{\varepsilon})\Delta\mathbf{u} + B\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{1}$$

- $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell)^T$ is a set of perturbation parameters; for simplicity of the presentation we assume that

$$\varepsilon_1 \leqslant \varepsilon_2 \leqslant \cdots \leqslant \varepsilon_\ell. \tag{2}$$

- B is a matrix-valued function, and $\mathbf{f}$ is a vector-valued function.

Let us assume that there is $\alpha_i > 0$ such that

$$b_{ii}(x) \geqslant \alpha_i^2 > 0; \tag{3a}$$

and that there are $\beta_i > 0$ such that

$$\beta_i^2 = \max_{\bar{\Omega}} \Big\{ b_{ii}(x)^{-1} \sum_{j \neq i} |b_{ij}(x)| \Big\}, \tag{3b}$$

and furthermore that

$$\beta_1\beta_2 < 1 \ \text{ if } \ell = 2 \qquad \text{and} \qquad \max_i \beta_i < 1 \text{ otherwise.} \tag{3c}$$

Define

$$\mathcal{L}_i u := -\varepsilon^2 \Delta u_i + b_{ii} u_i.$$

This (uncoupled) operator satisfies a Maximum Principle since the $b_{ii} > 0$. To extend this to a system:

Define a sequence of vector-valued functions $\mathbf{u}^{[k]}$ for $k = 0, 1, 2, \ldots$ as follows: let $\mathbf{u}^{[0]} = \mathbf{0}$ and for $k = 1, 2, \ldots,$ let $\mathbf{u}^{[k]}$ satisfy

$$L_i u_i^{[k]} = f_i - \sum_{j \neq i} b_{ij} u_j^{[k-1]} \qquad \text{on } \Omega,$$

$$u_i^{[k]}(x, y) = g_i(x, y) \qquad \text{on } \partial\Omega.$$

Then $\lim_{k\to\infty} \mathbf{u}^{[k]} = \mathbf{u}$.

If we now make the further assumption that

$$b_{ij} < 0 \qquad \text{for } i \neq j,$$

then

Lemma (Maximum Principle for systems)

If $L\mathbf{v} \geqslant \mathbf{0}$ *on* Ω *then* $\mathbf{v} \geqslant \mathbf{0}$ *on* $\bar{\Omega}$.

Proof.

The uncoupled operators satisfy a maximum principle.
And the right-hand side of each equation at each iteration is always non-negative. So... □

Now we construct a decomposition $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ of the solution of the system of equations (1), where, as before

- $\boldsymbol{v}$ is the regular solution component
- $\boldsymbol{w}$ represents the boundary layers.

Define $\kappa = \kappa(\zeta) > 0$ by

$$\kappa^2 := \min_i (1 - \beta_i) \min_k \alpha_k .$$

For arbitrary ε set

$$\mathcal{B}_\varepsilon(x) := e^{-\kappa x/\varepsilon} + e^{-\kappa(1-x)/\varepsilon} .$$

The solution decomposition is defined as follows. Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be the solutions of the boundary value problems

$$\mathcal{L}\boldsymbol{v} = \mathbf{f} \quad \text{in } (0,1), \;\; \boldsymbol{v}(0) = \mathbf{B}(0)^{-1}\mathbf{f}(0), \;\; \boldsymbol{v}(1) = \mathbf{B}(1)^{-1}\mathbf{f}(1), \tag{4a}$$

and

$$\mathcal{L}\boldsymbol{w} = 0 \quad \text{in } (0,1), \;\; \boldsymbol{w}(0) = -\boldsymbol{v}(0), \;\; \boldsymbol{w}(1) = -\boldsymbol{v}(1). \tag{4b}$$

Theorem ([Linß and Madden, 2009])

Let $\mathbf{A}$ *and* $\mathbf{f}$ *be twice continuously differentiable. Then the solution* $\mathbf{v}$ *and* $\mathbf{w}$ *of* (4) *satisfy*

$$\left\|v_i^{(k)}\right\| \leqslant C\left(1+\varepsilon_i^{2-k}\right), \qquad \text{for } k=0,1,\ldots,4, \quad i=1,\ldots,\ell, \tag{5a}$$

$$\left|w_i^{(k)}(x)\right| \leqslant C\sum_{m=i}^{\ell} \varepsilon_m^{-k}\mathcal{B}_{\varepsilon_m}(x) \qquad \text{for } k=0,1,2, \quad i=1,\ldots,\ell, \tag{5b}$$

and

$$\left|w_i^{(k)}(x)\right| \leqslant C\varepsilon_i^{2-k}\sum_{m=1}^{\ell} \varepsilon_m^{-2}\mathcal{B}_{\varepsilon_m}(x) \quad \text{for } k=3,4, \quad i=1,\ldots,\ell. \tag{5c}$$

We won't consider how to derive these bounds, but they give us some sense as to how to construct a suitable mesh.

We first look at the generalisation of the *Shishkin* mesh from earlier. Recall that this is a piecewise uniform mesh, adapted to the layer structure of the problem.

- Let N be divisible by $2(\ell+1)$.
- Let $\sigma > 0$ be arbitrary.
- Fix mesh transition points τ_k as follows

$$\tau_{\ell+1} = 1/2, \quad \tau_k = \min\left\{\frac{k\tau_{k+1}}{k+1}, \frac{\sigma\varepsilon_k}{\kappa}\ln N\right\}, \quad k = \ell, \dots, 1.$$

Then the mesh is obtained by dividing each of the intervals $[\tau_k, \tau_{k+1}]$ and $[1-\tau_{k+1}, 1-\tau_k]$, $k = 0, \dots, \ell$, into $N/(2\ell+2)$ subintervals of equal length.

Here σ relates to the formal order of the underlying discretization. For our finite difference scheme, $\sigma \geqslant 2$. Then

$$\|\mathbf{U} - \mathbf{u}\|_{\bar{\Omega}^N} \leqslant CN^{-2}\ln^2 N.$$

In practice, it can be easier to define the mesh in a way that is related to the pointwise bounds on the solution decomposition.

The most immediate way of doing this is to chose the mesh points such that

$$\int_0^{x_k} M_E(t)dt = \frac{k}{N}\int_0^1 M_E(t)dt \quad \text{with} \quad M_E(t) := 1 + \sum_{m=1}^{\ell} \varepsilon_m^{-1}\mathcal{B}_{2\varepsilon_m}(t),$$

i.e., the mesh equidistributes the monitor functions M_E. It can then be shown that

$$\|\mathbf{U} - \mathbf{u}\|_{\bar{\Omega}^N} \leqslant CN^{-2}.$$

That is, we can remove the spoiling logarithmic term associated with the Shishkin mesh.

However, constructing this mesh exactly requires that a nonlinear equation be solved for each mesh point x_i.

However, an approximate solution can be computed using a few iterations.

Bakhvalov meshes [Bakhvalov, 1969] can be considered as equidistributing the non-smooth monitor function

$$M_B(t) := \max\left\{1, \frac{q_1}{\varepsilon_1}e^{-\kappa t/\sigma\varepsilon_1}, \frac{q_1}{\varepsilon_1}e^{-\kappa(1-t)/\sigma\varepsilon_1}, \dots, \frac{q_\ell}{\varepsilon_\ell}e^{-\kappa t/\sigma\varepsilon_\ell}, \frac{q_\ell}{\varepsilon_\ell}e^{-\kappa(1-t)/\sigma\varepsilon_\ell}\right\}$$

with positive user chosen constants σ and q_m. For this mesh explicit formulae for the mesh points can be derived.

If $\sigma \geqslant 2$ then

$$\|\mathbf{U}-\mathbf{u}\|_{\bar{\Omega}^N} \leqslant CN^{-2},$$

Numerical Example

Example

$$
\begin{aligned}
-\varepsilon_1^2 u_1'' + 3u_1 + (1-x)(u_2 - u_3) &= e^x, & u_1(0) = u_1(1) = 0,\\
-\varepsilon_2^2 u_2'' + 2u_1 + (4+x)u_2 - u_3 &= \cos x, & u_2(0) = u_2(1) = 0,\\
-\varepsilon_3^2 u_3'' + 2u_1 + 3u_3 &= 1 + x^2, & u_3(0) = u_3(1) = 0.
\end{aligned}
$$

	Shishkin mesh		Bakhvalov mesh		equidistr. mesh	
N	η^N	p^N	η^N	ρ^N	η^N	ρ^N
$8 \cdot 2^3$	4.895e-02	1.22	4.910e-03	2.08	4.892e-03	2.20
$8 \cdot 2^4$	2.276e-02	1.50	1.157e-03	2.05	1.062e-03	1.90
$8 \cdot 2^5$	8.854e-03	1.67	2.790e-04	2.04	2.851e-04	2.01
$8 \cdot 2^6$	3.091e-03	1.77	6.771e-05	2.01	7.061e-05	2.00
$8 \cdot 2^7$	1.014e-03	1.83	1.676e-05	2.00	1.770e-05	2.00
$8 \cdot 2^8$	3.201e-04	1.88	4.179e-06	2.00	4.423e-06	2.00
$8 \cdot 2^9$	9.831e-05	1.91	1.044e-06	2.00	1.106e-06	2.00
$8 \cdot 2^{10}$	2.955e-05	1.94	2.610e-07	2.00	2.764e-07	2.00
$8 \cdot 2^{11}$	8.725e-06	1.96	6.527e-08	2.00	6.910e-08	2.00
$8 \cdot 2^{12}$	2.537e-06	—	1.632e-08	—	1.728e-08	—

References

Bakhvalov, N. (1969).
Towards optimization of methods for solving boundary value problems in the presence of boundary layers.
Zh. Vychisl. Mat. i Mat. Fiz., 9:841–859.

Kellogg, R., Madden, N., and Stynes, M. (2008).
A parameter-robust numerical method for a system of reaction-diffusion equations in two dimensions.
Numer. Methods Partial Differ. Equations, 24(1):312–334.

Linß, T. (2010).
Layer-adapted meshes for reaction-convection-diffusion problems, volume 1985 of *Lecture Notes in Mathematics*.
Springer-Verlag, Berlin.

Linß, T. and Madden, N. (2009).
Layer-adapted meshes for a linear system of coupled singularly perturbed reaction-diffusion problems.
IMA J. Numer. Anal., 29(1):109–125.

References

Linß, T. and Stynes, M. (2009).

Numerical solution of systems of singularly perturbed differential equations.

Comput. Methods Appl. Math., 9(2):165–191.

Madden, N. and Stynes, M. (2003).

A uniformly convergent numerical method for a coupled system of two singularly perturbed linear reactiondiffusion problems.

IMA Journal of Numerical Analysis, 23(4):627–644.

Shishkin, G. I. (1995).

Mesh Approximation of Singularly Perturbed Boundary-value Problems for Systems of Elliptic and Parabolic Equations.

Comp. Maths. Math. Phys., 35(4):429–446.