AARMS-CRM Workshop on NA of SPDEs, July 2016

http://www.math.mun.ca/~smaclachlan/anasc_spde/

Short course on Numerical Analysis of Singularly Perturbed Differential Equations

Niall Madden, NUI Galway (Niall.Madden@NUIGalway.ie)

§6 PDEs (ii): Elliptic problems in two dimensions

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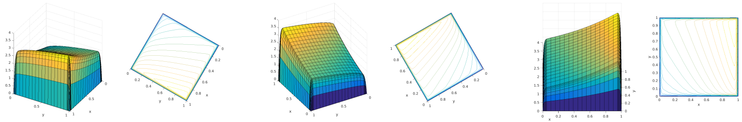


Handout version.

Outline

	Monday, 25 July	Tuesday, 26 July
09:00	Welcome/Coffee	
09:15	1. Introduction to singularly perturbed problems	5. PDEs (i): time-dependent problems.
10:00	Break	
10:15	2. Numerical methods and uniform convergence; FDMs and their analysis.	6. PDEs (ii): elliptic problems 7. Finite Element Methods
12:00	*Lunch*	
14:00	3. Coupled systems	8. Convection-diffusion (Stynes)
15:00	Break	
15:15	3. Coupled systems (continued)	9. Nonlinear problems (Kopteva)
16:15	4. Lab 1	10. Lab 2 (PDEs)
17:30	*Finish*	

§6. PDEs Part 2: Elliptic problems in two dimensions

(45 minutes)

In this section we will study the robust solution, by **a finite difference method**, of PDEs of the form

$$-\varepsilon^2 \Delta u + bu = f \qquad \text{on } \Omega := (0,1)^d.$$

The focus is on $d = 2$, but many of the ideas for $d = 3$ are similar.
(Come to my talk later in the week to learn about that case!).

Primary references

The key reference for this presentation is [Clavero et al., 2005]. From that, the most important component is the solution decomposition, which itself was first established by [Shishkin, 1992]. The compatibility conditions provided by [Han and Kellogg, 1990] are also vital.

Extensions to coupled systems can be found in [Kellogg et al., 2008a] and [Kellogg et al., 2008b], and a unified treatment is given in [Linß, 2010, Chap. 9].

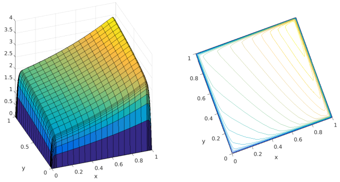
. . . .

The references above are mentioned only because they are related to the this presentation.

There are, of course, many other important papers on the solution of two-dimensional reaction-diffusion problems...

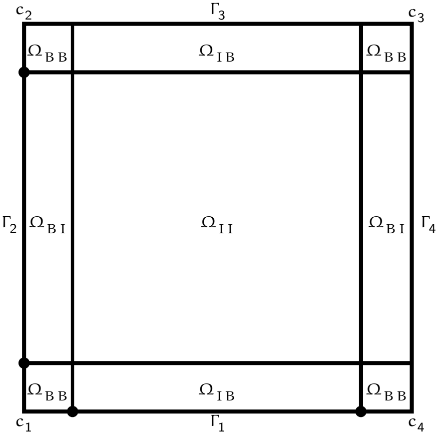
A 2D singularly perturbed problem

$$-\varepsilon^2(u_{xx}+u_{yy})+b(x,y)u=f(x,y), \text{ on } (0,1)^2 \qquad u=g \text{ on the boundary.} \tag{1}$$



- Typically, on this domain, solutions feature four "edge" layers that behave like $\exp(-x/\varepsilon)$ or $\exp(-y/\varepsilon)$.
- They also have four corner layers, that behave like $\exp(-(x+y)/\varepsilon)$.

- We'll denote the corners of the domain $c_1, \ldots, c_4$, labelled clockwise from $c_1 = (0, 0)$.
- The edges are Γ_1, ..., Γ_4, labelled clockwise from $\Gamma_1 = [0, 1]$.
- On the boundary, $u(x, y) = g(x, y)$, and we'll denote the restriction of g to Γ_i as g_i.



From [Han and Kellogg, 1990], we shall assume that $f, b \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$, the $g_i \in \mathcal{C}^{4,\alpha}([0,1])$ and that we have compatibility conditions at each corner. For example, at $c_1 = (0,0)$, these are

$$g_1 = g_2, \tag{2a}$$

$$-\varepsilon^2\Big(\frac{\partial^2}{\partial x^2} g_1 + \frac{\partial^2}{\partial y^2} g_2\Big) + b g_1 = f,^{1} \tag{2b}$$

$$\frac{\partial^2}{\partial x^2}\Big(-\varepsilon^2 \frac{\partial^2}{\partial x^2} g_1 + b g_1 - f\Big) = \frac{\partial^2}{\partial y^2}\Big(-\varepsilon^2 \frac{\partial^2}{\partial y^2} g_2 + b g_2 - f\Big). \tag{2c}$$

If u solves (1), and the conditions (2) are satisfied, as well as analogous ones at the other three corners, then $u \in \mathcal{C}^{4,\alpha}$.

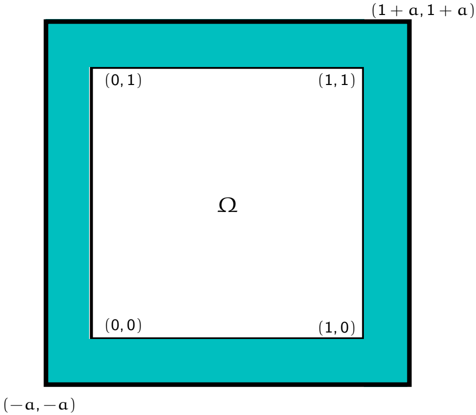
[1] Actually, g_1 and g_2 are functions of a single variable, x and y respctively, but it is notationally convienent to express these ordinary derivatives as partial derivaties, particularly in (2c).

One can show that

$$\left\|\frac{\partial^{(k+j)}}{\partial x^k \partial y^j} u\right\| \leqslant C\varepsilon^{-(k+j)}, \qquad (3)$$

but finer results are needed.

One of the key ideas in proving the existence of a suitable solution decomposition for this problem is to use an *extended domain*: $\Omega^\star = (-a, 1+a)^2$.
Define smooth extensions to b and f to $\bar{\Omega}^\star$, denoted $b^\star$ and $f^\star$ respectively.
Similarly the extension of g_i to $[-a, 1+a]$ is $g_i^\star$.



We will let $v^\star = v_0^\star + \varepsilon v_1^\star$, where

- $v_0^\star = f^\star / b^\star$.
- $v_1^\star$ solves

$$\mathcal{L}^\star v_1^\star = \Delta v_0^\star \quad \text{on } \Omega^\star, \qquad v_1^\star|_{\partial\Omega^\star} = 0.$$

- Then v is taken as the solution to

$$\mathcal{L}v = f \quad \text{on } \Omega^\star, \qquad v = v^\star \text{ on } \partial\Omega.$$

It follows that

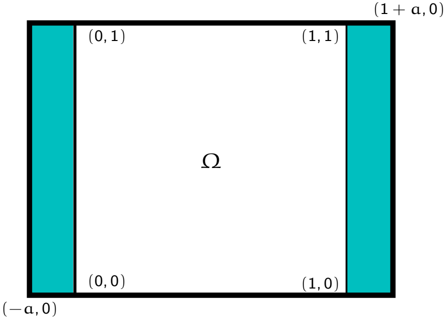
$$\left\| \frac{\partial^{(k+j)}}{\partial x^k \partial y^j} u \right\| \leqslant C(1 + \varepsilon^{-(k+j)}), \qquad \text{for } 0 \leqslant k + j \leqslant 4. \tag{4}$$

Next define a function w_1 which is associated with the edge along Γ_1. That is, we would like to construct w_1 so that

$$|w_1(x, y)| \leqslant Ce^{-\beta y/\varepsilon}.$$

Define a new extended domain, $\Omega^{\star\star} = (-a, 1 + a) \times (0, 1)$.
Let $w^\star$ solve

$$\begin{aligned}
\mathcal{L}^{\star\star} w_1 &= 0 && \text{on } \Omega^{\star\star}, \\
w_1^\star &= u - v && \text{on } \Gamma_1 \\
w_1^\star(x, 1) &= 0 && \text{for } x \in [-a, 1 + a], \\
w_1^\star(-a, y) &= 0 && \text{for } y \in [0, 1], \\
w_1^\star(1 + a, y) &= 0 && \text{for } y \in [0, 1],
\end{aligned}$$



and whatever conditions are needed on the remaining regions, $\big((-a, 0) \cup (1, 1 + a)\big) \times \{0\}$, to ensure that $w_1 \in C^{4,\alpha}(\bar{\Omega}^{\star\star})$. One can then show that

$$|w_1^\star(x, y)| \leqslant C\Big(\frac{a + x}{a}\Big)\Big(\frac{1 + a - x}{1 + a}\Big)e^{-\beta y/\varepsilon} \quad \text{for } (x, y) \in \bar{\Omega}^{\star\star}.$$

Next, define w_1 as the solution to

$$\begin{aligned} \mathcal{L}w_1 &= 0 && \text{on } \Omega, \\ w_1 &= u - v && \text{on } \Gamma_1 \\ w_1 &= 0 && \text{on } \Gamma_3 \\ w_1 &= w_1^\star && \text{on } \{0,1\} \times [0,1] \end{aligned}$$

Using the previous bound on $w_1^\star$ we get

$$|w_1(x,y)| \leqslant Ce^{-\beta y/\varepsilon} \qquad \text{for } (x,y) \in \bar{\Omega}.$$

So this shows that $w_1(x,y)$ decays rapidly away from Γ_1, the edge at $y = 1$.

It is possible establish analogous bounds for lower derivatives of w (more about that after coffee...).

Moreover, analogous bounds are possible for:

$$\begin{aligned} |w_2(x,y)| &\leqslant Ce^{-\beta x/\varepsilon} \\ |w_3(x,y)| &\leqslant Ce^{-\beta(1-y)/\varepsilon} \\ |w_4(x,y)| &\leqslant Ce^{-\beta(1-x)/\varepsilon} \end{aligned}$$

Finally, define z_1 (the component associated with the corner $c_1 = (0,0)$), as the solution to

$$\begin{aligned} \mathcal{L}z_1 &= 0 && \text{on } \Omega, \\ z_1 &= -w_2 && \text{on } \Gamma_1 \\ z_1 &= -w_1 && \text{on } \Gamma_2 \\ z_1 &= 0 && \text{on } \Gamma_3 \cup \Gamma_4 \end{aligned}$$

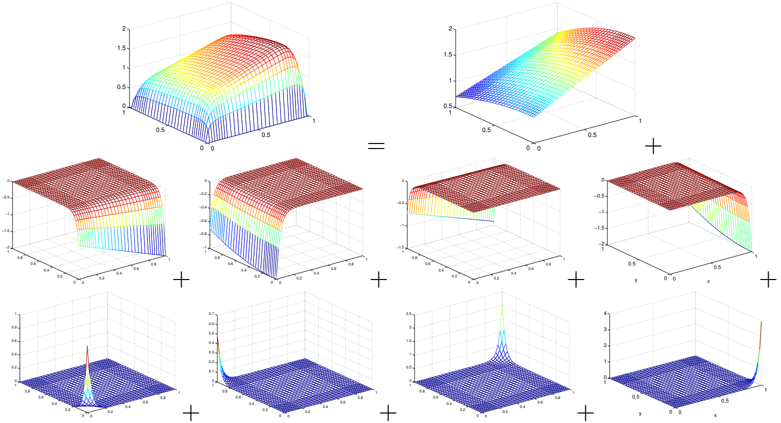
Since we have suitable compatibility conditions, $z_1 \in \mathcal{C}^{4,\alpha}$. A comparison principle then gives

$$|z_1(x,y)| \leqslant Ce^{-\beta(x+y)/\varepsilon}.$$

There are analogous functions, z_2, z_3 and z_4 associated with the other corners.

The decomposition is

$$u = v + \sum_{i=1}^{4} w_i + \sum_{i=1}^{4} z_i .$$



We re-use the finite difference method that we employed for one-dimensional problems, extended in the obvious way.

Let $\bar{\Omega}_x^N$ and $\bar{\Omega}_y^N$ be arbitrary meshes with N intervals on $[0,1]$.
Set $\bar{\Omega}^N = \{(x_i, y_j)\}_{i,j=0}^N$ to be the Cartesian product of $\bar{\Omega}_x^N$ and $\bar{\Omega}_y^N$.
Set $h_i = x_i - x_{i-1}$ and $k_i = y_i - y_{i-1}$ for each i.
Define the standard second-order central difference operators

$$\delta_x^2 v_{i,j} := \frac{1}{\bar{h}_i}\left(\frac{v_{i+1,j} - v_{i,j}}{h_{i+1}} - \frac{v_{i,j} - v_{i-1,j}}{h_i}\right)$$

$$\delta_y^2 v_{i,j} := \frac{1}{\bar{k}_i}\left(\frac{v_{i,j+1} - v_{i,j}}{k_{i+1}} - \frac{v_{i,j} - v_{i,j-1}}{k_i}\right)$$

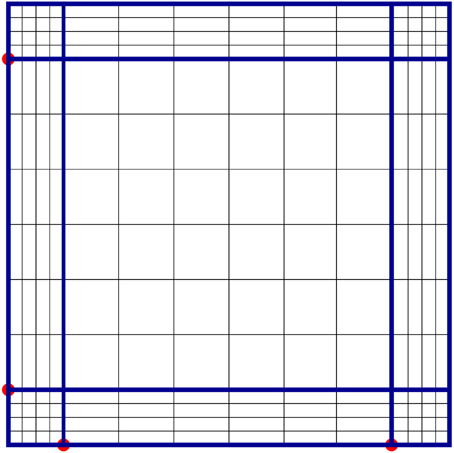
Define $\Delta^N v_{i,j} := (\delta_x^N + \delta_y^N) v_{i,j}$.
Then the difference operator is

$$(L^N U)_{i,j} = -\varepsilon^2 \Delta^N U_{i,j} + b(x_i, y_j) U_{i,j}, \quad i,j = 1, \dots N-1.$$

To generate a numerical approximation of the solution to (1) solve the system of $(N+1)^2$ linear equations

$$\begin{aligned} (L^N U)_{i,j} &= f(x_i, y_j) && \text{for } (x_i, y_j) \in \Omega^N, \\ U_{i,j} &= g(x_i, y_i) && \text{for } (x_i, y_j) \in \partial\Omega^N. \end{aligned} \qquad (5)$$

Define $\tau_\varepsilon = \min\left\{\frac{1}{4}, 2\frac{\varepsilon}{\beta}\ln N\right\}$, and construct $\bar{\Omega}_x^N$ and $\bar{\Omega}_y^N$ to be Shishkin meshes as before.



For the method and mesh, we would like to prove that

$$\|u - U\|_{\Omega^N} \leqslant C(N^{-1} \ln N)^2.$$

However, we shall show some restraint, and prove the easiest part of this: for the regular part.

But we will at least focus on how, without greatly complicating the analysis, we may show *almost second-order convergence*, compared to the first-order convergence we obtained for the scalar problem.

That is, assume there exists a decomposition of the discrete solution U:

$$U = V + \sum_{i=1}^{4} W_i + \sum_{i=1}^{4} Z_i.$$

We will just estimate $\|v - V\|_{\bar{\Omega}^N}$. The idea used is originally from [Miller et al., 1998], though the version given here is exactly from [Clavero et al., 2005].

Analysis (regular part only)

We need only a bound for the truncation error. Standard arguments give

$$|L^N(U-u)(x_i,y_j)| \leqslant \begin{cases} C\varepsilon^2\big(\bar{h}_i\|\frac{\partial^3}{\partial x^3}u\| + \bar{k}_j\|\frac{\partial^3}{\partial y^3}u\|\big) & x_i, y_j \in \{\tau_\varepsilon, 1-\tau\} \\ C\varepsilon^2\big(\bar{h}_i^2\|\frac{\partial^4}{\partial x^4}u\| + \bar{k}_j^2\|\frac{\partial^4}{\partial y^4}u\|\big) & \text{otherwise.} \end{cases}$$

From this

$$|L^N(V-v)(x_i,y_j)| \leqslant \begin{cases} C\varepsilon N^{-1} & x_i, y_j \in \{\tau_\varepsilon, 1-\tau\} \\ CN^{-2}. \end{cases}$$

Define the barrier function

$$\Phi(x_i,y_j) = C\frac{(\tau_\varepsilon)^2}{\varepsilon^2}N^{-2}\big(\Theta(x_i)+\Theta(y_j)\big) + CN^{-2},$$

where Θ is the piecewise linear function interpolating the points

$$\Big\{(0,0),(\tau_\varepsilon,1),(1-\tau_\varepsilon,1),(1,0)\Big\}$$

.

Then, for example,

$$\delta_x^2\Theta(x) = \begin{cases} -N/\tau_\varepsilon & x \in \{\tau_\varepsilon, 1-\tau_\varepsilon\} \\ 0 & \text{otherwise.} \end{cases}$$

Analysis (regular part only)

It follows directly that

$$0 \leqslant \Phi(x_i, y_i) \leqslant CN^{-2} \ln^2 N,$$

and

$$|L^N \Phi(x_i, y_j)| \leqslant \begin{cases} C\tau_{\varepsilon} N^{-1} + (b\Phi)(x_i, y_j) & x_i, y_j \in \{\tau_{\varepsilon}, 1-\tau\} \\ (b\Phi)(x_i, y_j) & \text{otherwise.} \end{cases}$$

Application of a maximum principle gives

$$\|v - V\|_{\bar{\Omega}^N} \leqslant CN^{-2} \ln^2 N.$$

....

The remaining analysis for $\|w_i - W_i\|_{\bar{\Omega}^N}$ and $\|z_i - Z_i\|_{\bar{\Omega}^N}$ is quite involved, and the details are not presented here.

However, in the next section of this short course, we'll look at the analysis of such terms when studying a *finite element method*.

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