AARMS-CRM Workshop on NA of SPDEs, July 2016

http://www.math.mun.ca/~smaclachlan/anasc_spde/

Short course on Numerical Analysis of Singularly Perturbed Differential Equations

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§7 Finite Element Methods

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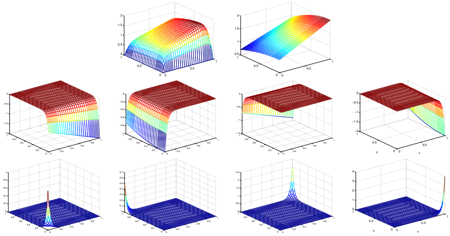


Handout version.

Outline

	Monday, 25 July	Tuesday, 26 July
09:00	Welcome/Coffee	
09:15	1. Introduction to singularly perturbed problems	5. PDEs (i): time-dependent problems.
10:00	Break	
10:15	2. Numerical methods and uniform convergence; FDMs and their analysis.	6. PDEs (ii): elliptic problems 7. Finite Element Methods
12:00	*Lunch*	
14:00	3. Coupled systems	8. Convection-diffusion (Stynes)
15:00	Break	
15:15	3. Coupled systems (continued)	9. Nonlinear problems (Kopteva)
16:15	4. Lab 1	10. Lab 2 (PDEs)
17:30	*Finish*	

(60 minutes)

In this section we will study the analysis of a **finite element method**, of PDEs of the form

$$-\varepsilon^2 \Delta u + bu = f \qquad \text{on } \Omega := (0,1)^d.$$

We will use a standard Galerkin method on a tensor product space with bilinear elements, on a Shishkin mesh (again!). Analysing the method, we'll obtain an error estimate that is *parameter robust*, in the sense that dependence on ε is entirely accounted for.
However, the estimate is not independent of ε, we will finish with a discussion of appropriate norms for this problem.

Primary references

The main reference to this section is [Liu et al., 2009]. Although that article is primarily about a *sparse grid method*, it also provides a sharp analysis of a standard Galerkin FEM.

Again, we'll rely on the solution decomposition whose exposition was presented in [Clavero et al., 2005].

If you would like to read some more about FEMs+SPPS, a good starting point would be [Linß and Madden, 2004], which has a simple analysis of a system of two coupled reaction-diffusion problems in one-dimension, on Shishkin and Bakhvalov meshes.

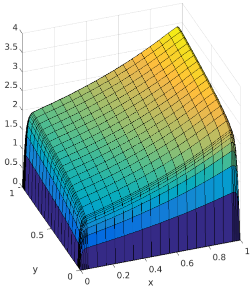
As usual, the monograph [Linß, 2010] gives a more detailed analysis, including sections on quadrature, etc. See also [Roos et al., 2008].

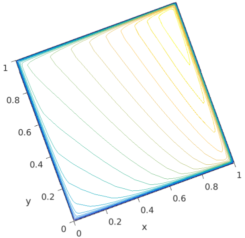
The more recent material, on balanced norms, is motivated by [Lin and Stynes, 2012], and the discussion in [Adler et al., 2016].

A 2D, SP, reaction-diffusion equation

A 2D singularly perturbed problem

$$-\varepsilon^2(u_{xx}+u_{yy})+b(x,y)u=f(x,y), \text{ on } \Omega:=(0,1)^2 \qquad u|_{\partial\Omega}=0. \qquad (1)$$





As before, we expect the solution to exhibit 9 distinct regions: the interior, four edge layer regions, and four corner layer regions.

A 2D singularly perturbed problem

$$-\varepsilon^2(u_{xx}+u_{yy})+b(x,y)u=f(x,y),\ \text{on } \Omega:=(0,1)^2 \qquad u|_{\partial}\Omega=0. \tag{2}$$

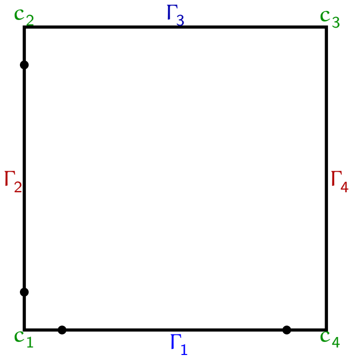
As usual, $\varepsilon \in (0,1]$, but also $b(x,y) \geqslant 2\beta^2 > 0$.

We assume that $f,\ b \in C^{4,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1]$. It follows that $u \in C^{6,\alpha}(\Omega)$. We also assume that f vanishes at each corner of $\bar{\Omega}$ to ensure that $u \in C^{3,\alpha}(\bar{\Omega})$.

The edges of $\partial\Omega$ are

$$\Gamma_1 := \{(x,0)|0 \leqslant x \leqslant 1\}, \quad \Gamma_2 := \{(0,y)|0 \leqslant y \leqslant 1\},$$
$$\Gamma_3 := \{(x,1)|0 \leqslant x \leqslant 1\}, \quad \Gamma_4 := \{(1,y)|0 \leqslant y \leqslant 1\}.$$

Label the corners of $\bar{\Omega}$ as c_1, c_2, c_3, c_4 where c_1 is $(0,0)$ and the numbering is clockwise.



Again, we make use of the Shishkin decomposition from [Clavero et al., 2005], with minor variations.

Subject to the assumptions that $b, f \in C^{4,\alpha}(\bar{\Omega})$, and corner compatibility conditions, the solution u can be decomposed as

$$u = v + w + z = v + \sum_{i=1}^{4} w_i + \sum_{i=1}^{4} z_i,$$

where each w_i is a layer associated with the edge Γ_i, and each z_i is a layer associated with the corner c_i. There exists a constant C such that

$$\left|\frac{\partial^{m+n} v}{\partial x^m \partial y^n}(x,y)\right| \leqslant C(1+\varepsilon^{2-m-n}), \qquad 0 \leqslant m+n \leqslant 4,$$

$$\left|\frac{\partial^{m+n} w_1}{\partial x^m \partial y^n}(x,y)\right| \leqslant C(1+\varepsilon^{2-m})\varepsilon^{-n} e^{-\beta y/\varepsilon} \qquad 0 \leqslant m+n \leqslant 3,$$

$$\left|\frac{\partial^{m+n} w_2}{\partial x^m \partial y^n}(x,y)\right| \leqslant C(1+\varepsilon^{2-n})\varepsilon^{-m} e^{-\beta x/\varepsilon} \qquad 0 \leqslant m+n \leqslant 3,$$

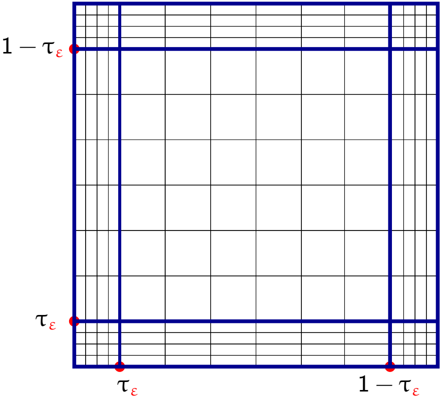
$$\left|\frac{\partial^{m+n} z_1}{\partial x^m \partial y^n}(x,y)\right| \leqslant C\varepsilon^{-m-n} e^{-\beta(x+y)/\varepsilon} \qquad 0 \leqslant m+n \leqslant 3,$$

with analogous bounds for w_3, w_4, z_2, z_3 and z_4.

The Shishkin mesh

We use the same Shishkin mesh as for the Finite Difference method. Define

$$\tau_\varepsilon = \min\left\{\frac{1}{4},\ 2\varepsilon\beta^{-1}\ln N\right\}.$$



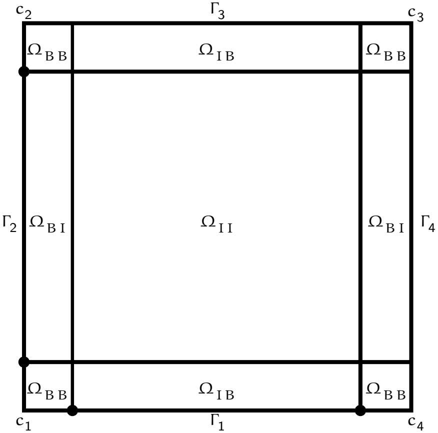
The Shishkin mesh

We will consider the case where ε is so small that

$$\tau_\varepsilon = 2\varepsilon\beta^{-1}\ln N.$$

Partition Ω as follows: $\bar{\Omega} = \Omega_{II} \cup \Omega_{BI} \cup \Omega_{IB} \cup \Omega_{BB}$, where

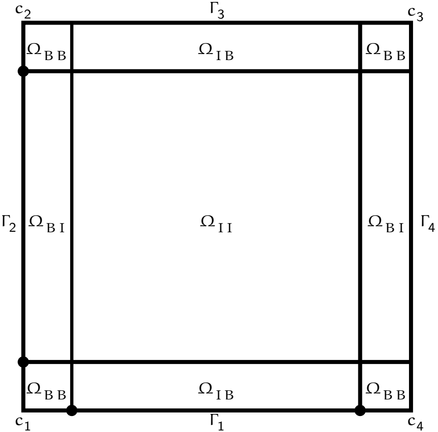
$$\begin{aligned}
\Omega_{II} &= [\tau_\varepsilon, 1-\tau_\varepsilon] \times [\tau_\varepsilon, 1-\tau_\varepsilon],\\
\Omega_{BI} &= ([0,\tau_\varepsilon] \cup [1-\tau_\varepsilon, 1]) \times [\tau_\varepsilon, 1-\tau_\varepsilon],\\
\Omega_{IB} &= [\tau_\varepsilon, 1-\tau_\varepsilon] \times ([0,\tau_\varepsilon] \cup [1-\tau_\varepsilon, 1]),\\
\Omega_{BB} &= ([0,\tau_\varepsilon] \times ([0,\tau_\varepsilon] \cup [1-\tau_\varepsilon, 1]))\\
&\quad \cup ([1-\tau_\varepsilon, 1] \times ([0,\tau_\varepsilon] \cup [1-\tau_\varepsilon, 1])).
\end{aligned}$$



In the case of interest, $\varepsilon \leqslant N^{-1}$, and so $\tau = 2\varepsilon\beta^{-1}\ln N$. Thus, for any point $(x, y) \in \Omega_{II}$,

$$e^{-\beta x/\varepsilon} \leqslant e^{-\beta\tau/\varepsilon} = N^{-2},$$

$$e^{-\beta y/\varepsilon} \leqslant e^{-\beta\tau/\varepsilon} = N^{-2}.$$



$$\|e^{-\beta(x+y)/\varepsilon}\|_{0,\Omega/\Omega_{BB}} \leqslant \frac{\varepsilon}{\beta}N^{-2};$$

$$\|e^{-\beta(x+y)/\varepsilon}\|_{0,\Omega_{BB}} = \frac{\varepsilon}{2\beta}.$$

$$\|e^{-\beta y/\varepsilon}\|^2_{0,\Omega_{II}\cup\Omega_{BI}} = \|e^{-\beta x/\varepsilon}\|^2_{0,\Omega_{II}\cup\Omega_{IB}} \leqslant \frac{\varepsilon}{2\beta}N^{-4}.$$

$$\|e^{-\beta y/\varepsilon}\|^2_{0,\Omega_{BB}\cup\Omega_{IB}} = \|e^{-\beta x/\varepsilon}\|^2_{0,\Omega_{BB}\cup\Omega_{BI}} \leqslant \frac{\varepsilon}{2\beta}.$$

Given a one-dimensional mesh, Ω_x^N, let V^N be the associated space of piecewise linear functions.

Let $I^N : C[0,1] \to V^N[0,1]$ be the usual piecewise linear Lagrange interpolation operator associated with V^N.

Let $p \in [2,\infty]$ and $\phi \in W^{2,p}[0,1]$. Then the piecewise linear interpolant $I_N\phi$ of ϕ satisfies the bounds

$$\|\phi - I_N\phi\|_{0,p,[x_{i-1},x_i]} + h_i\|(\phi - I_N\phi)'\|_{0,p,[x_{i-1},x_i]} \leqslant \\ C\min\left\{h_i\|\phi'\|_{0,p,[x_{i-1},x_i]},\ h_i^2\|\phi''\|_{0,p,[x_{i-1},x_i]}\right\}.$$

From standard inverse inequalities in one dimension one sees easily that

$$h_x\left\|\frac{\partial\psi}{\partial x}\right\|_{0,K} + k_y\left\|\frac{\partial\psi}{\partial y}\right\|_{0,K} \leqslant \|\psi\|_{0,K} \quad \forall\psi \in V^{N_x,N_y}(\Omega), \quad \forall K \in T^{N_x,N_y}(\Omega), \tag{3}$$

where the rectangle K has size $h_x \times k_y$.

Interpolation

The Shishkin mesh is highly anisotropic on $\Omega_{IB} \cup \Omega_{BI}$, and to obtain satisfactory interpolation error estimates on this region one uses the sharp anisotropic interpolation analysis of [Apel, 1999, Apel and Dobrowolski, 1992]:

Lemma

Let τ be any mesh rectangle of size $h_x \times k_y$. Let $\phi \in H^2(\tau)$. Then its piecewise bilinear nodal interpolant ϕ^I satisfies the bounds

$$\|\phi - \phi^I\|_{0,\tau} \leqslant C\left(h_x^2\|\phi_{xx}\|_{0,\tau} + h_x k_y\|\phi_{xy}\|_{0,\tau} + k_y^2\|\phi_{yy}\|_{0,\tau}\right),$$

$$\|(\phi - \phi^I)_x\|_{0,\tau} \leqslant C\left(h_x\|\phi_{xx}\|_{0,\tau} + k_y\|\phi_{xy}\|_{0,\tau}\right),$$

$$\|(\phi - \phi^I)_y\|_{0,\tau} \leqslant C\left(h_x\|\phi_{xy}\|_{0,\tau} + k_y\|\phi_{yy}\|_{0,\tau}\right).$$

Equipped with these results, we would like to prove that

Lemma

There exists a constant C *such that*

$$\|u - I_{N,N}u\|_{0,\Omega} \leqslant CN^{-2}. \tag{4a}$$

and

$$\varepsilon\|\nabla(u - I_{N,N}u)\|_{0,\Omega} \leqslant C(N^{-2} + \varepsilon^{1/2}N^{-1}\ln N). \tag{4b}$$

Here we will give an account of how the bound for the (4b) term is obtained.

From the solution decomposition,

$$\varepsilon\|\nabla(u - I_{N,N}u)\|_{0,\Omega} = \varepsilon\left\|\nabla\left((I - I_{N,N})\left(v + \sum_{k=1}^{4} w_k + \sum_{k=1}^{4} z_k\right)\right)\right\|_{0,\Omega}.$$

Each term in this decomposition is bounded separately.

First, standard arguments give,

$$\varepsilon\left\|\frac{\partial}{\partial x}(v - I_{N,N}v)\right\|_{0,\Omega} \leqslant C\varepsilon N^{-1}|v|_{2,\Omega} \leqslant CN^{-2}.$$

Interpolation

Recall that w_1 is the term associated with Γ_1, and, so $w_1(x,y) \sim e^{-y\beta/\varepsilon}$. That fact, and the anisotropic interpolation results, give

$$\begin{aligned}
\varepsilon \left\| \frac{\partial}{\partial x}(w_1 - I_{N,N} w_1) \right\|_{0,\Omega_{II}\cup\Omega_{BI}} &\leqslant C\varepsilon N^{-1} \left(\left\| \frac{\partial^2 w_1}{\partial x^2} \right\|_{0,\Omega_{II}\cup\Omega_{BI}} \right. \\
&\qquad \left. + \left\| \frac{\partial^2 w_1}{\partial x \partial y} \right\|_{0,\Omega_{II}\cup\Omega_{BI}} \right) \\
&\leqslant C\varepsilon N^{-1} \left(1 + \max_{(x,y)\in\Omega_{II}\cup\Omega_{BI}} \varepsilon^{-1} e^{-\beta y/\varepsilon} \right) \\
&\leqslant C N^{-2}.
\end{aligned}$$

Interpolation

On $\Omega_{IB} \cup \Omega_{BB}$,

$$\varepsilon \left\| \frac{\partial}{\partial x}(w_1 - I_{N,N} w_1) \right\|_{0,\Omega_{IB} \cup \Omega_{BB}} \leqslant C\varepsilon \left[N^{-1} \left\| \frac{\partial^2 w_1}{\partial x^2} \right\|_{0,\Omega_{IB} \cup \Omega_{BB}} + \varepsilon N^{-1} (\ln N) \left\| \frac{\partial^2 w_1}{\partial x \partial y} \right\|_{0,\Omega_{IB} \cup \Omega_{BB}} \right] \leqslant CN^{-2}.$$

Thus $\varepsilon \left\| \frac{\partial}{\partial x}(w_1 - I_{N,N} w_1) \right\|_{0,\Omega} \leqslant CN^{-2}$.

Next recall that w_2 is the component associated with the edge layer near Γ_2. So, roughly, $w_2(x, y) \sim e^{-x\beta/\varepsilon}$.

Similar to above, we can show that $\varepsilon\|(w_2 - I_{N,N}w_2)_x\|_{0,\Omega_{II}\cup\Omega_{IB}} \leqslant N^{-2}$.

However, the most significant term is

$$\varepsilon \left\| \frac{\partial}{\partial x}(w_2 - I_{N,N}w_2) \right\|_{0,\Omega_{BI}\cup\Omega_{BB}} \leqslant$$
$$C\varepsilon \left[\varepsilon N^{-1}(\ln N) \left\| \frac{\partial^2 w_2}{\partial x^2} \right\|_{0,\Omega_{BI}\cup\Omega_{BB}} + N^{-1} \left\| \frac{\partial^2 w_2}{\partial x \partial y} \right\|_{0,\Omega_{BI}\cup\Omega_{BB}} \right]$$
$$\leqslant C\varepsilon^{1/2}N^{-1}\ln N.$$

Consequently,

$$\varepsilon \left\| \frac{\partial}{\partial x}(w_2 - I_{N,N}w_2) \right\|_{0,\Omega} \leqslant C(N^{-2} + \varepsilon^{1/2}N^{-1}\ln N).$$

Analogous results are valid for w_3, w_4 and the corner layer terms, z_1, z_2, z_3, z_4.
Gathering these results yields $\varepsilon \left\| \frac{\partial}{\partial x}(u - I_{N,N}u) \right\|_{0,\Omega} \leqslant C(N^{-2} + \varepsilon^{1/2}N^{-1} \ln N)$.
The same estimate is valid for $\varepsilon \| \frac{\partial}{\partial y}(u - I_{N,N}u) \|_{0,\Omega}$.
It is then clear that

Theorem

There exists a constant C *such that*

$$\|u - I_{N,N}u\|_{0,\Omega} + \varepsilon \|\nabla(u - I_{N,N}u)\|_{0,\Omega} \leqslant C(N^{-2} + \varepsilon^{1/2}N^{-1} \ln N).$$

The Galerkin FEM

The variational formulation of (2) is: find $u \in H_0^1(\Omega)$ such that

$$B(u, v) := \varepsilon^2(\nabla u, \nabla v) + (bu, v) = (f, v) \quad \forall v \in H_0^1(\Omega).$$

Define an associated *energy norm*

$$\|v\|_\varepsilon := \left\{\varepsilon^2\|\nabla v\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2\right\}^{1/2}.$$

This bilinear form is coercive with respect to this norm:

$$B(v, v) = \varepsilon^2 \left\|\frac{\partial v}{\partial x}\right\|_{0,\Omega}^2 + \varepsilon^2 \left\|\frac{\partial v}{\partial y}\right\|_{0,\Omega}^2 + b\|v\|_{0,\Omega}^2 \geqslant \min\{1, 2\beta^2\} \|v\|_\varepsilon^2 \quad \forall v \in H_0^1(\Omega).$$

Furthermore it is continuous

$$|B(v, w)| \leqslant (2 + \|b\|_{0,\infty,\Omega})\|v\|_\varepsilon\|w\|_\varepsilon \quad \forall v, w \in H_0^1(\Omega).$$

The Galerkin FEM

Define the Galerkin finite element approximation $u_{N,N} \in V_0^{N,N}(\Omega)$

$$B(u_{N,N}, v_{N,N}) = (f, v_{N,N}) \quad \forall v_{N,N} \in V_0^{N,N}(\Omega).$$

Classical finite element arguments based on coercivity and Galerkin orthogonality yields the quasioptimal bound

$$\|u - u_{N,N}\|_\varepsilon \leqslant C \inf_{\phi \in V_0^{N,N}(\Omega)} \|u - \phi\|_\varepsilon \leqslant \|u - I_{N,N}u\|_\varepsilon.$$

It then follows that...

Theorem

There exists a constant C *such that*

$$\|u - u_{N,N}\|_\varepsilon \leqslant C(N^{-2} + \varepsilon^{1/2}N^{-1}\ln N).$$

Numerical Example

Example

$$-\varepsilon^2 \Delta u + \big(1 + x^2 y^2 e^{xy/2}\big) u = f \quad \text{on } \Omega := (0,1)^2,$$

where f and the boundary conditions are chosen so that

$$u = x^3(1+y^2) + \sin(\pi x^2) + \cos(\pi y/2) + (x+y)\big(e^{-2x/\varepsilon} + e^{-2(1-x)/\varepsilon} + e^{-3y/\varepsilon} + e^{-3(1-y)/\varepsilon}\big).$$

ε^2	$N = 2^4$	$N = 2^6$	$N = 2^8$	$N = 2^{10}$
1	3.395e-1	8.714e-2	2.190e-2	5.482e-3
10^{-2}	4.618e-1	1.572e-1	4.214e-2	1.070e-2
10^{-4}	2.287e-1	1.578e-1	7.228e-2	2.510e-2
10^{-6}	7.220e-2	4.979e-2	2.280e-2	7.921e-3
10^{-8}	2.361e-2	1.574e-2	7.211e-3	2.504e-3
10^{-10}	9.621e-3	4.992e-3	2.280e-3	7.919e-4
10^{-12}	6.787e-3	1.619e-3	7.214e-4	2.504e-4
10^{-14}	6.435e-3	6.265e-4	2.292e-4	7.920e-5

Other norms

The above results are somewhat suspect looking... although the method does resolve layers, the error, in both theory and practice, shows an ε-dependency.

However, it is observed that (subject to sufficient regularlity),

$$\|u - u_{N,N}\|_{\infty,\bar{\Omega}^N} \leqslant CN^{-2}.$$

So, in some sense, the difficulty is with the norm, rather than the method.

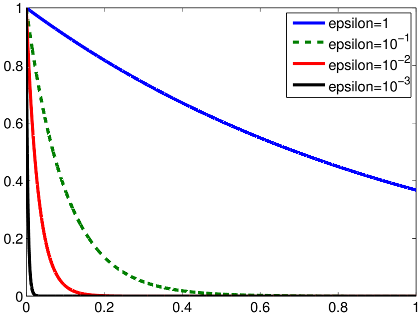
Consider this very simple one-dimensional singularly perturbed reaction-diffusion problem:

$$-\varepsilon^2 u''(x) + u(x) = 0 \text{ on } (0,1),$$

$$u(0) = 1, u(1) = e^{-1/\epsilon} (\approx 0).$$

Its solution is $u(x) = e^{-x/\epsilon}$.

$\varepsilon = \{1, 10^{-1}, 10^{-2}, 10^{-3}\}$



$$\|u\|_\infty := \max_{0\leqslant x} |u(x)| = 1, \quad \text{but} \quad \|u\|_0 := \sqrt{\int_0^1 (u(x))^2 dx} \approx \sqrt{\varepsilon}.$$

As $\varepsilon \to 0$, we get that $\|u\|_0 \to 0$, even though $\|u\|_\infty \to 1$.

Trivially, this shows that $u^h \equiv 0$ is a terrible approximation to u with respect to $\|\cdot\|_\infty$, but rather good with respect to $\|\cdot\|_0$.

Slightly less trivially, try solving this problem with a standard Galerkin FEM. The weak form is:

$$B(u, v) := \int_0^1 \varepsilon^2 u'(x)v'(x) + u(x)v(x), \qquad (f, v) := \int_0^1 f(x)v(x),$$

and find $u \in H_0^1(0, 1)$.

$$B(u, v) = (f, v) \quad \text{for all } v \in H_0^1(0, 1).$$

The energy norm is

$$\|g\|_{\varepsilon} := \left(\varepsilon^2 \|g'\|_0^2 + \|g\|_0^2\right)^{1/2}.$$

But this norm is weak, since

$$\left(\varepsilon^2 \|u'\|_0 + \|u\|_0\right)^{1/2} \approx \sqrt{\varepsilon}.$$

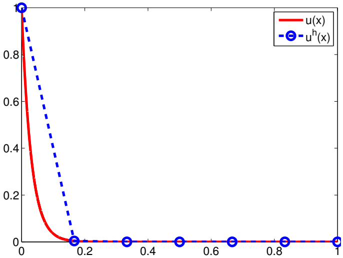
In contrast,

$$\left(\varepsilon \|u'\|_2 + \|u\|_2\right)^{1/2} \approx 1.$$

Suppose we did try to solve our simple ODE with a Galerkin FEM with linear elements on a uniform mesh... Clearly, even though this is a "good" estimate at mesh points, it is clear that

$$\|u - u_N\|_{\infty,\Omega} \sim \mathcal{O}(1).$$

Approximation with $\varepsilon = 10^{-2}$



It is known ([Bagaev and Shaĭdurov, 1998], [Farrell et al., 2000]) that

$$\|u - u^N\|_{\varepsilon} \leqslant CN^{-1/2}.$$

So we now have two problems with the energy norm:

- it appears to show robust convergence even when layers are not being resolved.
- On the layer-resolving Shishkin mesh, the $\mathcal{O}(\varepsilon^{1/2}N^{-1}\ln N)$ quantity demonstrates that this norm is not "balanced".

There are several approaches to resolving the problem of the weakness of the usual energy norm for this problem:

(a) Analyse a standard FEM (on a suitable mesh), but with respect to a stronger norm, such as

$$\|v\|_{\mathsf{bal}} := \left(\varepsilon\|\nabla v\|_0^2 + \|v\|_0^2\right)^{1/2}.$$

This is done in [Roos and Schopf, 2014], and also [Melenk and Xenophontos, 2015].

(b) Design a new FEM for which the natural induced norm *is* balanced. E.g.,
- In [Lin and Stynes, 2012], this is done using a first-order system approach.
- In FOSLS-type setting, see [Adler et al., 2016]
- In [Roos and Schopf, 2014], a C^0 interior penalty (CIP) method is constructed.

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