AARMS-CRM Workshop on NA of SPDEs, July 2016

http://www.math.mun.ca/~smaclachlan/anasc_spde/

Short course on Numerical Analysis of Singularly Perturbed Differential Equations

Niall Madden

§0 Outline

Version 24.07.16

Outline

	Monday, 25 July	Tuesday, 26 July
09:00	Welcome/Coffee	
09:15	1. Introduction to singularly perturbed problems	5. PDEs (i): time-dependent problems.
10:00	Break	
10:15	2. Numerical methods and uniform convergence; FDMs and their analysis.	6. PDEs (ii): elliptic problems 7. Finite Element Methods
12:00	*Lunch*	
14:00	3. Coupled systems	8. Convection-diffusion (Stynes)
15:00	Break	
15:15	3. Coupled systems (continued)	9. Nonlinear problems (Kopteva)
16:15	4. Lab 1	10. Lab 2 (PDEs)
17:30	*Finish*	

Outline

Credits and disclaimers

The following notes were prepared by Niall Madden for a short course on the numerical analysis of singularly perturbed differential equations.

Some of the material is original, but much of it is derived from research papers and key texts, including

- Linß: *Layer-adapted meshes for reaction-convection-diffusion problems.* 2010. [1].
- Miller, O'Riordan, Shishkin: *Fitted numerical methods for singular perturbation problems*. 2012. [2]
- O'Malley: *Think about ordinary differential equations*. 1997 [3]
- Roos, Stynes and Tobiska,: *Robust numerical methods for singularly perturbed differential equations. Convection-diffusion-reaction and flow problems.* 2008. [4]
- Shishkin and Shishkina: *difference methods for singular perturbation problems*. 2009. [5]
- Stynes: *Steady-state convection-diffusion problems*. Acta Numerica 2005. [6]
- Etc.

§1. Introduction to singularly perturbed problems

(45 minutes)

1.1 When is a perturbation singular?

1.2 Singularly Perturbed DEs

1.3 The Bestiary

1.4 Reaction-diffusion equations ODEs

1.5 Convection-diffusion equations ODEs

1.6 Coupled systems

1.7 Reaction-diffusion PDEs

1.8 Convection-diffusion PDEs

1.9 Other problems

1.10 Discussion

1.11 References

§2. Numerical methods for SPDEs

(1:45)

2.1 The need for special schemes for reaction-diffusion problems.

2.2 The need for special schemes for convection-diffusion problems.

2.3 Uniform convergence and layer-resolving methods

2.4 Maximum principles and stability.

2.5 Bound on derivatives.

2.6 Solution decompositions.

2.7 A Finite difference method on a layer-adapted (Shishkin) mesh.

2.8 Numerical Analysis

§3. Coupled systems

(1 hour)

3.1 Overview of coupled systems

3.2 A coupled systems of two equations, with interacting boundary layers.

3.3 A finite difference method and a Shishkin mesh

3.4 Analysis: solution decompositions.

3.5 Generalisations to larger systems: stability and decompositions.

3.6 Graded meshes.

§4. Lab 1

(90 minutes)

4.1 Implement a standard central difference technique for a single equation: verify that analysis from Section 2 is not sharp.

4.2 Convection-diffusion problems: stability.

4.3 Convection-diffusion problems: choice of mesh.

4.4 Coupled systems.

§5. PDEs (i): time-dependent problems

(45 minutes)

5.1 Solving $u_t + \mathcal{L}u = f$, where $\mathcal{L}$ is a singularly perturbed operator.

5.2 Discretization: central differences and implicit time-stepping.

5.3 The stationary problem:

- discrete Green's functions;
- stability;
- error analysis.

5.4 The time-dependent problem.

5.5 Various layer-adapted meshes.

. . . .

§6. PDEs (ii): elliptic problems in two dimensions

(45 minutes)

In this section we will study the robust solution, by finite difference methods, of PDEs of the form

$$-\varepsilon^2 \Delta u + bu = f \qquad \text{on } \Omega := (0,1)^d.$$

The focus is on $d = 2$, but many of the ideas for $d = 3$ are similar. (Come to my talk later in the week to learn about that case!).

6.1 A linear reaction-diffusion equation in two dimensions.

6.2 Solution decomposition: compatibility conditions; the *extended domain* technique; edge and corner laters.

6.3 Discretization.

6.4 Proving almost second-order convergence.

§7. Finite element methods

(1 hour)

7.1 The two-dimensional reaction diffusion problem again.

7.2 That decomposition, again.

7.3 That Shishkin mesh, again.

7.4 Interpolation results.

7.5 Some numerical results ...

7.6 ... norms.

. . . .

§8. Convection-Diffusion problems (Stynes)

(1 hour)

8.1 Nature of solutions to convection-diffusion problems in one and two dimensions.

8.2 Why are they difficult to solve numerically? Central differencing versus upwinding

8.3 Numerical methods: exponential fitting, SDFEM, Shishkin meshes

. . . .

§9. Nonlinear problems (Kopteva)

(45 minutes)

§10. Lab 2

(90 minutes?)

2.5.1 A finite difference method in 2D.

2.5.2 Compatibility conditions.

2.5.3 Graded meshes.

Bibliography

T. Linß.
Layer-adapted meshes for reaction-convection-diffusion problems, volume 1985 of *Lecture Notes in Mathematics*.
Springer-Verlag, Berlin, 2010.

J. J. H. Miller, E. O'Riordan, and G. I. Shishkin.
Fitted numerical methods for singular perturbation problems.
World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, revised edition, 2012.

Robert E. O'Malley, Jr.
Thinking about ordinary differential equations.
Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1997.

Hans-Gorg Roos, Martin Stynes, and Lutz Tobiska.
Robust Numerical Methods for Singularly Perturbed Differential Equations, volume 24 of *Springer Series in Computational Mathematics*.
Springer-Verlag, Berlin, 2nd edition, 2008.

Bibliography

Grigory I. Shishkin and Lidia P. Shishkina.
Difference methods for singular perturbation problems, volume 140 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*.
CRC Press, Boca Raton, FL, 2009.

Martin Stynes.
Steady-state convection-diffusion problems.
Acta Numer., 14:445–508, 2005.