Theory & Computation of Singularly Perturbed Differential Equations

IIT (BHU) Varanasi, Dec 2017

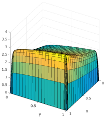
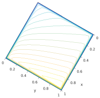
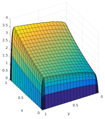
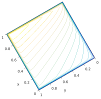
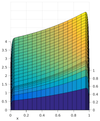
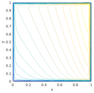
https://skumarmath.wordpress.com/gian-17/singular-perturbation-problems/

http://www.maths.nuigalway.ie/~niall/TCSPDEs2017

Niall Madden, NUI Galway

§8 Singularly perturbed elliptic PDEs

Version 06.12.17

Handout version.

Outline

Monday, 4 December		
09:30 – 10.30	Registration and Inauguration	
10:45 – 11.45	1. Introduction to singularly perturbed problems	NM
12:00 – 13:00	2. Numerical methods and uniform convergence	NM
14:30 – 15:30	Tutorial (Convection diffusion problems)	NM
15:30 – 16:30	Lab 1 (Simple FEMs in MATLAB)	NM

Tuesday, 5 December		
09:30 – 10:30	3. Finite difference methods and their analyses	NM
10:45 – 11:45	4. Coupled systems of SPPDEs	NM
14:00 – 16:00	Lab 2 (Fitted mesh methods for ODEs)	NM

Thursday, 7 December		
09:00 – 10:00	8. Singularly perturbed elliptic PDEs	NM
10:15 – 11:15	9. Finite Elements in two and three dimensions	NM
01:15 – 15:15	Lab 4 (Singularly perturbed PDEs)	NM

Friday, 8 December		
09:00 – 10:00	10. Preconditioning for SPPs	NM

§8. Singular Perturbed Elliptic Problems

($\approx$ 1 hour)

In this section we will study the robust solution, by **a finite difference method**, of PDEs of the form

$$-\varepsilon^2 \Delta u + bu = f \qquad \text{on } \Omega := (0,1)^d.$$

The focus is on $d = 2$, but many of the ideas for $d = 3$ are similar, which will be mentioned in the next section.

Primary references

The key reference for this presentation is [Clavero et al., 2005]. From that, the most important component is the solution decomposition, which itself was first established by [Shishkin, 1992]. The compatibility conditions provided by [Han and Kellogg, 1990] are also vital.

Extensions to coupled systems can be found in [Kellogg et al., 2008a] and [Kellogg et al., 2008b], and a unified treatment is given in [Linß, 2010, Chap. 9].

The most general treatment is given in [Shishkin and Shishkina, 2009], though it is not the easiest book to read.

. . . .

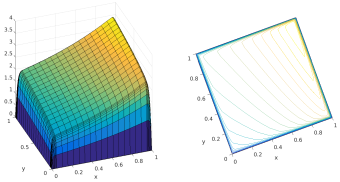
The references above are mentioned only because they are related to the this presentation.

There are, of course, many other important papers on the solution of two-dimensional reaction-diffusion problems...

A 2D, SP, reaction-diffusion equation

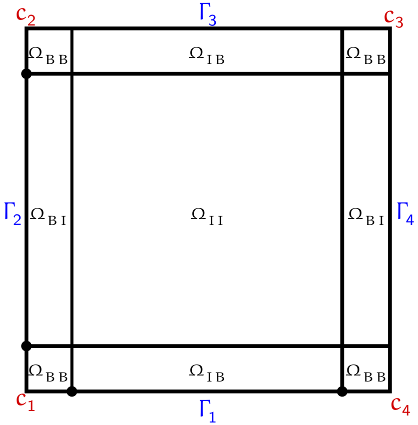
A 2D singularly perturbed problem

$$-\varepsilon^2(u_{xx} + u_{yy}) + b(x,y)u = f(x,y), \text{ on } \Omega := (0,1)^2$$
$$u = g \text{ on } \partial\Omega. \tag{1}$$



- Typically, on this domain, solutions feature four "edge" layers that behave like $\exp(-x/\varepsilon)$ or $\exp(-y/\varepsilon)$.
- They also have four corner layers, that behave like $\exp(-(x+y)/\varepsilon)$.

- We'll denote the corners of the domain $c_1, \ldots, c_4$, labelled clockwise from $c_1 = (0, 0)$.
- The edges are Γ_1, . . . , Γ_4, labelled clockwise from $\Gamma_1 = [0, 1]$.
- $u(x, y) = g(x, y)$ on $\partial\Omega$, and g_i is the restriction of g to Γ_i.



From [Han and Kellogg, 1990], we shall assume that $f, b \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$, the $g_i \in \mathcal{C}^{4,\alpha}([0,1])$ and that we have compatibility conditions at each corner. For example, at $c_1 = (0,0)$, these are

$$g_1 = g_2, \tag{2a}$$

$$-\varepsilon^2\Big(\frac{\partial^2}{\partial x^2}g_1 + \frac{\partial^2}{\partial y^2}g_2\Big) + bg_1 = f,^1 \tag{2b}$$

$$\frac{\partial^2}{\partial x^2}\Big(-\varepsilon^2\frac{\partial^2}{\partial x^2}g_1 + bg_1 - f\Big) = \frac{\partial^2}{\partial y^2}\Big(-\varepsilon^2\frac{\partial^2}{\partial y^2}g_2 + bg_2 - f\Big). \tag{2c}$$

If u solves (1), and the conditions (2) are satisfied, as well as analogous ones at the other three corners, then $u \in \mathcal{C}^{4,\alpha}$.

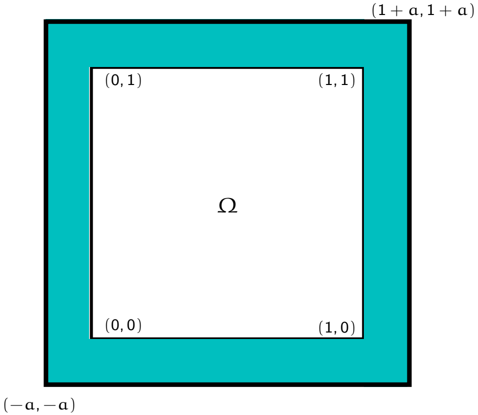
[1] Actually, g_1 and g_2 are functions of a single variable, x and y respsctively, but it is notationally convienent to express these ordinary derivatives as partial derivaties, particularly in (2c).

One can show that

$$\left\| \frac{\partial^{(k+j)}}{\partial x^k \partial y^j} u \right\| \leqslant C\varepsilon^{-(k+j)}, \qquad \text{for } k, j \in \mathbb{N}_0, k + j \leqslant 4, \tag{3}$$

but finer results are needed.

One of the key ideas in proving the existence of a suitable solution decomposition for this problem is to use an *extended domain*:
$\Omega^\star = (-a, 1 + a)^2$.
Define smooth extensions to b and f to $\bar{\Omega}^\star$, denoted $b^\star$ and $f^\star$ respectively. Similarly the extension of g_i to $[-a, 1 + a]$ is $g_i^\star$.



We will let $v^\star = v_0^\star + \varepsilon v_1^\star$, where

- $v_0^\star = f^\star/b^\star$.
- $v_1^\star$ solves
$$\mathcal{L}^\star v_1^\star = \Delta v_0^\star \quad \text{on } \Omega^\star, \qquad v_1^\star|_{\partial\Omega^\star} = 0.$$
- Then v is taken as the solution to
$$\mathcal{L}v = f \quad \text{on } \Omega^\star, \qquad v = v^\star \text{ on } \partial\Omega.$$

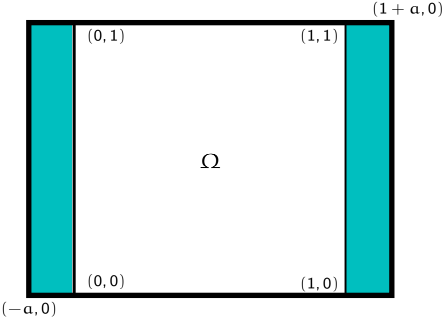
It follows that

$$\|\frac{\partial^{(k+j)}}{\partial x^k \partial y^j} u\| \leqslant C(1 + \varepsilon^{-(k+j)}), \qquad \text{for } 0 \leqslant k+j \leqslant 4. \tag{4}$$

Next define a function w_1 which is associated with the edge along Γ_1.
That is, we would like to construct w_1 so that $|w_1(x,y)| \leqslant Ce^{-\beta y/\varepsilon}$.

Define a new extended domain,
$\Omega^{\star\star} = (-a, 1+a) \times (0,1)$.
Let $w^\star$ solve

$$
\begin{aligned}
\mathcal{L}^{\star\star} w_1 &= 0 && \text{on } \Omega^{\star\star},\\
w_1^\star &= u - v && \text{on } \Gamma_1\\
w_1^\star(x,1) &= 0 && \text{for } x \in [-a, 1+a],\\
w_1^\star(-a,y) &= 0 && \text{for } y \in [0,1],\\
w_1^\star(1+a,y) &= 0 && \text{for } y \in [0,1],
\end{aligned}
$$



and whatever conditions are needed on the remaining regions, $\big((-a,0) \cup (1,1+a)\big) \times \{0\}$, to ensure that $w_1 \in C^{4,\alpha}(\bar{\Omega}^{\star\star})$. One can then show that

$$|w_1^\star(x,y)| \leqslant C\Big(\frac{a+x}{a}\Big)\Big(\frac{1+a-x}{1+a}\Big)e^{-\beta y/\varepsilon} \quad \text{for } (x,y) \in \bar{\Omega}^{\star\star}.$$

Next, define w_1 as the solution to

$$\begin{aligned} \mathcal{L}w_1 &= 0 && \text{on } \Omega, \\ w_1 &= u - v && \text{on } \Gamma_1 \\ w_1 &= 0 && \text{on } \Gamma_3 \\ w_1 &= w_1^\star && \text{on } \{0,1\} \times [0,1] \end{aligned}$$

Using the previous bound on $w_1^\star$ we get

$$|w_1(x,y)| \leqslant Ce^{-\beta y/\varepsilon} \quad \text{for } (x,y) \in \bar{\Omega}.$$

So this shows that $w_1(x,y)$ decays rapidly away from Γ_1, the edge at $y = 1$.

It is possible establish analogous bounds for lower derivatives of w (more about that after coffee...).

Moreover, analogous bounds are possible for:

$$|w_2(x,y)| \leqslant Ce^{-\beta x/\varepsilon} \qquad |w_3(x,y)| \leqslant Ce^{-\beta(1-y)/\varepsilon}$$

$$|w_4(x,y)| \leqslant Ce^{-\beta(1-x)/\varepsilon}$$

Finally, define z_1 (the component associated with the corner $c_1 = (0,0)$), as the solution to

$$\begin{aligned} \mathcal{L}z_1 &= 0 && \text{on } \Omega, \\ z_1 &= -w_2 && \text{on } \Gamma_1 \\ z_1 &= -w_1 && \text{on } \Gamma_2 \\ z_1 &= 0 && \text{on } \Gamma_3 \cup \Gamma_4 \end{aligned}$$

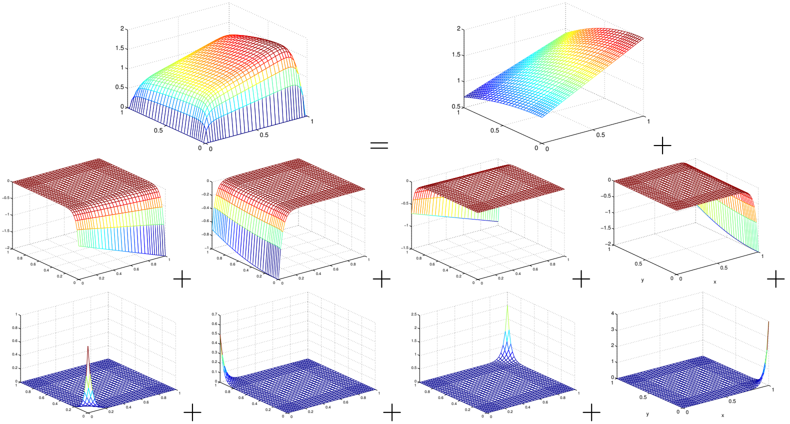
Since we have suitable compatibility conditions, $z_1 \in \mathcal{C}^{4,\alpha}$. A comparison principle then gives

$$|z_1(x,y)| \leqslant Ce^{-\beta(x+y)/\varepsilon}.$$

There are analogous functions, z_2, z_3 and z_4 associated with the other corners.

The decomposition is

$$u = v + \sum_{i=1}^{4} w_i + \sum_{i=1}^{4} z_i.$$



We re-use the finite difference method that we employed for 1D problems, extended in the obvious way.

Let $\bar{\Omega}_x^N$ and $\bar{\Omega}_y^N$ be arbitrary meshes with N intervals on $[0, 1]$.
Set $\bar{\Omega}^N = \{(x_i, y_j)\}_{i,j=0}^N$ to be the Cartesian product of $\bar{\Omega}_x^N$ and $\bar{\Omega}_y^N$.
Set $h_i = x_i - x_{i-1}$ and $k_i = y_i - y_{i-1}$ for each i.
Define the standard second-order central difference operators

$$\delta_x^2 v_{i,j} := \frac{1}{\bar{h}_i}\left(\frac{v_{i+1,j} - v_{i,j}}{h_{i+1}} - \frac{v_{i,j} - v_{i-1,j}}{h_i}\right)$$

$$\delta_y^2 v_{i,j} := \frac{1}{\bar{k}_i}\left(\frac{v_{i,j+1} - v_{i,j}}{k_{i+1}} - \frac{v_{i,j} - v_{i,j-1}}{k_i}\right)$$

Define $\Delta^N v_{i,j} := (\delta_x^N + \delta_y^N) v_{i,j}$.
Then the difference operator is

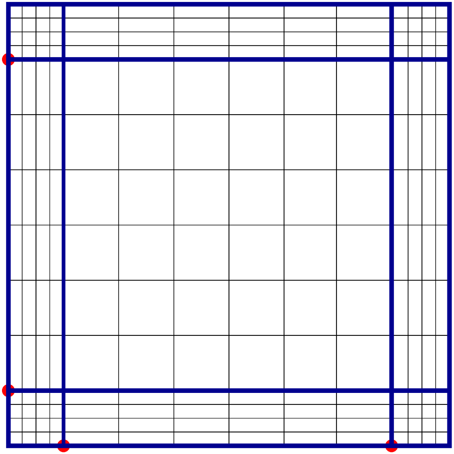
$$(L^N U)_{i,j} = -\varepsilon^2 \Delta^N U_{i,j} + b(x_i, y_j) U_{i,j}, \quad i, j = 1, \dots N-1.$$

To generate a numerical approximation of the solution to (1) solve the system of $(N+1)^2$ linear equations

$$\begin{aligned} (L^N U)_{i,j} &= f(x_i, y_j) && \text{for } (x_i, y_j) \in \Omega^N, \\ U_{i,j} &= g(x_i, y_i) && \text{for } (x_i, y_j) \in \partial\Omega^N. \end{aligned} \tag{5}$$

Solving such linear systems is interesting and challenging. It will be the topic of Lecture 10 tomorrow.

Define $\tau_\varepsilon = \min\left\{\frac{1}{4}, 2\frac{\varepsilon}{\beta}\ln N\right\}$, and construct $\bar{\Omega}_x^N$ and $\bar{\Omega}_y^N$ to be Shishkin meshes as before.



For the method and mesh, we would like to prove that

$$\|u - U\|_{\Omega^N} \leqslant C(N^{-1}\ln N)^2.$$

However, we shall show some restraint, and prove the easiest part of this: for the regular part.

But we will at least focus on how, without greatly complicating the analysis, we may show *almost second-order convergence*, compared to the first-order convergence we obtained for the scalar problem.

That is, assume there exists a decomposition of the discrete solution U:

$$U = V + \sum_{i=1}^{4} W_i + \sum_{i=1}^{4} Z_i.$$

We will just estimate $\|v - V\|_{\bar{\Omega}^N}$. The idea used is originally from [Miller et al., 1998], though the version given here is exactly from [Clavero et al., 2005].

Analysis (regular part only)

We need only a bound for the truncation error. Standard arguments give

$$|L^N(U - u)(x_i, y_j)| \leqslant \begin{cases} C\varepsilon^2\left(\bar{h}_i\|\frac{\partial^3}{\partial x^3}u\| + \bar{k}_j\|\frac{\partial^3}{\partial y^3}u\|\right) & x_i, y_j \in \{\tau_\varepsilon, 1-\tau\} \\ C\varepsilon^2\left(\bar{h}_i^2\|\frac{\partial^4}{\partial x^4}u\| + \bar{k}_j^2\|\frac{\partial^4}{\partial y^4}u\|\right) & \text{otherwise.} \end{cases}$$

From this

$$|L^N(V - v)(x_i, y_j)| \leqslant \begin{cases} C\varepsilon N^{-1} & x_i, y_j \in \{\tau_\varepsilon, 1-\tau\} \\ CN^{-2}. & \end{cases}$$

Define the barrier function

$$\Phi(x_i, y_j) = C\frac{(\tau_\varepsilon)^2}{\varepsilon^2}N^{-2}\big(\Theta(x_i) + \Theta(y_j)\big) + CN^{-2},$$

where Θ is the piecewise linear function interpolating the points

$$\Big\{(0,0), (\tau_\varepsilon, 1), (1-\tau_\varepsilon, 1), (1, 0)\Big\}.$$

Analysis (regular part only)

Define the barrier function

$$\Phi(x_i, y_j) = C\frac{(\tau_\varepsilon)^2}{\varepsilon^2}N^{-2}\big(\Theta(x_i) + \Theta(y_j)\big) + CN^{-2},$$

where Θ is the piecewise linear function interpolating the points

$$\Big\{(0,0), (\tau_\varepsilon, 1), (1-\tau_\varepsilon, 1), (1,0)\Big\}.$$

Then, for example,

$$\delta_x^2\Theta(x) = \begin{cases} -N/\tau_\varepsilon & x \in \{\tau_\varepsilon, 1-\tau_\varepsilon\} \\ 0 & \text{otherwise.} \end{cases}$$

Analysis (regular part only)

It follows directly that

$$0 \leqslant \Phi(x_i, y_i) \leqslant CN^{-2} \ln^2 N,$$

and

$$|L^N \Phi(x_i, y_j)| \leqslant \begin{cases} C\tau_\varepsilon N^{-1} + (b\Phi)(x_i, y_j) & x_i, y_j \in \{\tau_\varepsilon, 1-\tau\} \\ (b\Phi)(x_i, y_j) & \text{otherwise.} \end{cases}$$

Application of a maximum principle gives

$$\|v - V\|_{\bar{\Omega}^N} \leqslant CN^{-2} \ln^2 N.$$

. . . .

The remaining analysis for $\|w_i - W_i\|_{\bar{\Omega}^N}$ and $\|z_i - Z_i\|_{\bar{\Omega}^N}$ is quite involved, and the details are not presented here.

However, in the next section of this short course, we'll look at the analysis of such terms when studying a *finite element method*.

References

Clavero, C., Gracia, J. L., and O'Riordan, E. (2005).
A parameter robust numerical method for a two dimensional reaction-diffusion problem.
Math. Comp., 74(252):1743–1758.

Han, H. and Kellogg, R. B. (1990).
Differentiability properties of solutions of the equation $-\epsilon^2 \Delta u + ru = f(x, y)$ in a square.
SIAM J. Math. Anal., 21(2):394–408.

Kellogg, R., Madden, N., and Stynes, M. (2008a).
A parameter-robust numerical method for a system of reaction-diffusion equations in two dimensions.
Numer. Methods Partial Differ. Equations, 24(1):312–334.

References

Kellogg, R. B., Linss, T., and Stynes, M. (2008b).
A finite difference method on layer-adapted meshes for an elliptic reaction-diffusion system in two dimensions.
Math. Comp., 77(264):2085–2096.

Linß, T. (2010).
Layer-adapted meshes for reaction-convection-diffusion problems, volume 1985 of *Lecture Notes in Mathematics*.
Springer-Verlag, Berlin.

Miller, J. J. H., O'Riordan, E., Shishkin, G. I., and Shishkina, L. P. (1998).
Fitted mesh methods for problems with parabolic boundary layers.
Math. Proc. R. Ir. Acad., 98A(2):173–190.

References

Shishkin, G. I. (1992).
Discrete Approximation of Singularly Perturbed Elliptic and Parabolic Equations.
Russian Academy of Sciences, Ural Section, Ekaterinburg.
In Russian.

Shishkin, G. I. and Shishkina, L. P. (2009).
Difference methods for singular perturbation problems, volume 140 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*.
CRC Press, Boca Raton, FL.