Theory & Computation of Singularly Perturbed Differential Equations

IIT (BHU) Varanasi, Dec 2017

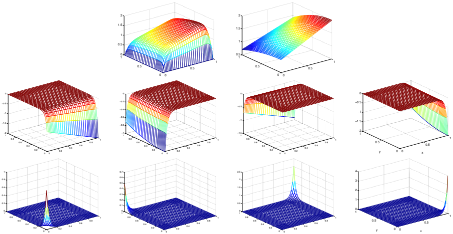
https://skumarmath.wordpress.com/gian-17/singular-perturbation-problems/

http://www.maths.nuigalway.ie/~niall/TCSPDEs2017

Niall Madden, NUI Galway

§9 Finite element methods for singularly perturbed PDEs

Version 06.12.17



Outline

Monday, 4 December		
09:30 – 10.30	Registration and Inauguration	
10:45 – 11.45	1. Introduction to singularly perturbed problems	NM
12:00 – 13:00	2. Numerical methods and uniform convergence	NM
14:30 – 15:30	Tutorial (Convection diffusion problems)	NM
15:30 – 16:30	Lab 1 (Simple FEMs in MATLAB)	NM

Tuesday, 5 December		
09:30 – 10:30	3. Finite difference methods and their analyses	NM
10:45 – 11:45	4. Coupled systems of SPPDEs	NM
14:00 – 16:00	Lab 2 (Fitted mesh methods for ODEs)	NM

Thursday, 7 December		
09:00 – 10:00	8. Singularly perturbed elliptic PDEs	NM
10:15 – 11:15	9. Finite Elements in two and three dimensions	NM
01:15 – 15:15	Lab 4 (Singularly perturbed PDEs)	NM

Friday, 8 December		
09:00 – 10:00	10. Preconditioning for SPPs	NM

§9. Finite Element Methods for SPPDEs

(≈ 60 minutes)
In this lecture we will study the analysis of a **finite element method**, of PDEs of the form

$$-\varepsilon^2 \Delta u + bu = f \quad \text{on } \Omega := (0,1)^d,$$

for $d = 1, 2, 3$. We will use a standard Galerkin method on a tensor product space with bilinear elements, on a Shishkin mesh (again!). We will analyse the method to obtain an error estimate that is *parameter robust*, in the sense that dependence on ε is entirely accounted for. However, the estimate is not independent of ε, we will finish with a discussion of appropriate norms for this problem.

Primary references

The main reference to this section is [Liu et al., 2009]. Although that article is primarily about a *sparse grid method*, it also provides a sharp analysis of a standard Galerkin FEM.

Again, we'll rely on the *solution decomposition* whose exposition was presented in [Clavero et al., 2005].

This talk is about 2D and 3D problems. If you **prefer 1D**, see [Linß and Madden, 2004], which has a simple analysis of a system of two coupled reaction-diffusion problems on Shishkin and Bakhvalov meshes.

As usual, the monograph [Linß, 2010] gives a more detailed analysis, including sections on quadrature, etc. See also [Roos et al., 2008].

The more recent material on **balanced norms**, is motivated by [Lin and Stynes, 2012], and the discussion in [Adler et al., 2016].

The details on 3D problems are based on [Russell and Madden, 2017a], and rely on a decomposition given in [Shishkin and Shishkina, 2009].

In addition, full **MATLAB source code** for a 2D (non-SPP) problem is available from `https://github.com/niallmadden/SparseGrids/`. See also, [Russell and Madden, 2017b].

FEM-101

The key sequence of ideas for FEMs is

(i) First replace the differential equation with an integral equation, using integration by parts to reduce the order of the derivatives. This is called the "*variational*" or

(ii) If we had a candidate for the true solution to the differential equation, we could trial it by substituting it back into the **BVP**.

(iii) But the set of possible solutions is infinitely large, so we can't check them all.

(iv) So we choose a much smaller subset, and look for the solution there. The space we will use is the space of *piecewise linear splines*.

(v) For every value of the spline that we have to determine, we write down a version of the integral equation that must be satisfied.

(vi) This will give us a linear system of equations to solve.

(vii) A simple but clever idea shows that the approximation we find is the best possible one.

First we write the BVP as an integral equation.

Define the inner product: $(u, v) := \int_a^b u(x)v(x)dx$.

Take the boundary Value Problem: *find* $u \in C^2(a, b)$ *such that*

$$-u''(x) + b(x)u(x) = f(x) \quad \text{on } (a, b),$$
$$u(a) = u(b) = 0.$$

Multiply by an arbitrary function v, and integrate by parts to get

Definition (Variational formulation)

The variational/weak formulation is: *Find* $u \in H_0^1(a, b)$ *such that*

$$\mathcal{A}(u, v) = L(v) \quad \textit{for all} \quad v \in H_0^1(a, b). \tag{1}$$

where $\mathcal{A}(\cdot, \cdot)$ is the (symmetric) bilinear functional

$$\mathcal{A}(u, v) := (u', v') + (ru, v).$$

and $L(v)$ is the linear functional $L(v) = (f, v)$.

The above problem is still not tractable. We would have trial every $u \in H^1_0(a, b)$, and test it against every $v \in H^1_0(a, b)$.

Since $H^1_0(a, b)$ is infinite dimensional, that is not feasible. So we choose a smaller subspace of $H^1_0(a, b)$.

First fix a "*mesh*" on $[a, b]$. This is just a set of points $\{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$. Then consider the space of all functions that are *piecewise linear* on this mesh and that vanish at $x = a$ and $x = b$.

This is a finite-dimensional sub-space of $H^1_0(a, b)$. A reasonable basis for this space would be the hat functions $\{\psi_1, \psi_2, \dots, \psi_{n-1}\}$ given by

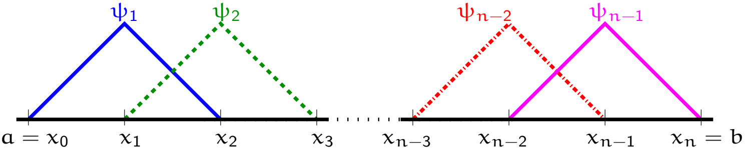
$$\psi_i(x) = \begin{cases} (x - x_{i-1})/h & x_{i-1} \leqslant x < x_i \\ (x_{i+1} - x)/h & x_i \leqslant x \leqslant x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

where $h = (b - a)/n$ is the distance between adjacent points.

Then we can write any function u_h as

$$u_h(x) = \lambda_1\psi_1(x) + \lambda_2\psi_2(x) + \cdots + \lambda_{n-1}\psi_{n-1}(x).$$

This basis set, shown below, are often called *hat functions* or *Galerkin basis functions*. We met them before in Section 2.1 on piecewise linear interpolation.



Definition (The Finite Element Method)

Let S be the finite dimensional subspace of $H_0^1(a, b)$ made up of the piecewise linear functions on a fixed mesh $a = x_0 < x_1 < \cdots < x_n = b$. Then the *Galerkin Finite Element method* is: *find* $u_h \in S$ *such that*

$$\mathcal{A}(u_h, v_h) = (f, v_h) \qquad \text{for all} \qquad v_h \in S. \tag{2}$$

We now want to look at how to turn this definition into an algorithm.

Let S be the space of piecewise linear functions on the mesh $x_i = a + ih$, where $h = (b-a)/n$. As above, u_h is can be written as

$$u_h(x) = \lambda_1\psi_1(x) + \lambda_2\psi_2(x) + \cdots + \lambda_{n-1}\psi_{n-1}(x).$$

So u_h has $n-1$ unknowns: $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$.

To solve for these, we need $n-1$ equations. To get these, we just choose $n-1$ different (i.e., linearly independent) possible v_h, and substitute into (2).

The most obvious, and (it turns out) sensible, choice for these $n-1$ equations are the $n-1$ hat functions ψ_1, $\psi_2, \dots, \psi_{n-1}$.

This gives us $n-1$ equations to solve:

$$\mathcal{A}(u_h, \psi_i) = (f, \psi_i) \qquad \text{for } i = 1, \dots n-1. \tag{3}$$

It is not difficult to see that, if we write these equations as a matrix-vector equation, $Ax = F$, then

$$a_{i,j} = \mathcal{A}(\psi_i, \psi_j)$$

It is easily proved that the member of S found by the FEM is the "closest" to the true solution.

Lemma (Cea's Lemma; Thm 14.6 of Süli and Mayers)

Let u be the solution to (1), i.e., the true solution, and let u_h be the solution to (2), i.e, the FE approximation.

(i) *Galerkin Orthogonality*

$$\mathcal{A}(u - u_h, v_h) = 0 \text{ for all } v_h \in S,$$

(ii) *There is no element of S that is closer to u than u_h:*

$$\mathcal{A}(u - u_h, u - u_h) = \min_{v_h \in S} \mathcal{A}(u - v_h, u - v_h),$$

The bilinear form $\mathcal{A}(\cdot,\cdot)$ induces the *norm:* $\|u\|_\varepsilon := \sqrt{\mathcal{A}(u,u)}$. So we can write (ii) of Cea's Lemma as

$$\|u - u_h\|_\varepsilon \leqslant |||u - v_h||| \qquad \text{for all } v_h \in S.$$

To turn this result into an error choose an function in S that we know is close to u, for example, its piecewise linear interpolant $I_N u$.

Then, the above optimality result gives

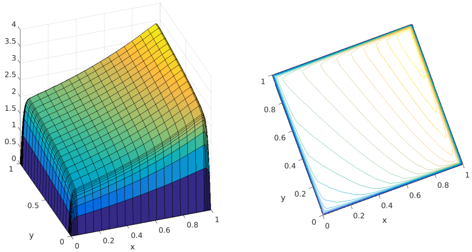
$$\|u - u_h\|_\varepsilon \leqslant \|u - I_N u\|_\varepsilon.$$

So the analysis reduces to a problem in classical approximation theory.

2D reaction-diffusion

A 2D singularly perturbed problem

$$-\varepsilon^2(u_{xx}+u_{yy})+b(x,y)u=f(x,y), \text{ on } \Omega:=(0,1)^2 \qquad (4)$$
$$u=g \text{ on } \partial\Omega.$$



As before, we expect the solution to exhibit 9 distinct regions: the interior, four edge layer regions, and four corner layer regions.

2D reaction-diffusion

A 2D singularly perturbed problem

$$-\varepsilon^2(u_{xx}+u_{yy})+b(x,y)u=f(x,y),\ \text{on } \Omega:=(0,1)^2$$
$$u=g \text{ on } \partial\Omega.$$

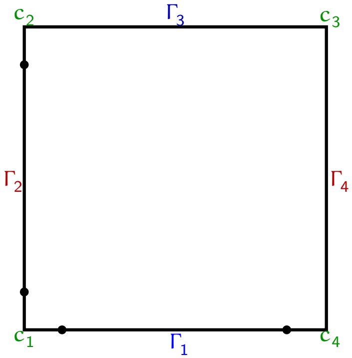
As usual, $\varepsilon \in (0,1]$, but also $b(x,y) \geqslant 2\beta^2 > 0$.

We assume that $f,\ b \in C^{4,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1]$. It follows that $u \in C^{6,\alpha}(\Omega)$. We also assume that f vanishes at each corner of $\bar{\Omega}$ to ensure that $u \in C^{3,\alpha}(\bar{\Omega})$.

The edges of $\partial\Omega$ are

$$\Gamma_1 := \{(x,0) | 0 \leqslant x \leqslant 1\}, \quad \Gamma_2 := \{(0,y) | 0 \leqslant y \leqslant 1\},$$
$$\Gamma_3 := \{(x,1) | 0 \leqslant x \leqslant 1\}, \quad \Gamma_4 := \{(1,y) | 0 \leqslant y \leqslant 1\}.$$

Label the corners of $\bar{\Omega}$ as c_1, c_2, c_3, c_4 where c_1 is $(0,0)$ and the numbering is clockwise.



Again, we use the Shishkin decomposition from [Clavero et al., 2005], with minor variations. Subject to the assumptions that $b, f \in C^{4,\alpha}(\bar{\Omega})$, and corner compatibility conditions, the solution u can be decomposed as

$$u = v + w + z = v + \sum_{i=1}^{4} w_i + \sum_{i=1}^{4} z_i,$$

where each w_i is a layer-type term associated with the edge Γ_i, and each z_i is a layer associated with the corner c_i.

$$u = v + w + z = v + \sum_{i=1}^{4} w_i + \sum_{i=1}^{4} z_i.$$

Each w_i is a layer-type term associated with the edge Γ_i, and each z_i is a layer associated with the corner c_i.

There exists a constant C such that

$$\left|\frac{\partial^{m+n} v}{\partial x^m \partial y^n}(x,y)\right| \leqslant C(1+\varepsilon^{2-m-n}), \qquad 0 \leqslant m+n \leqslant 4,$$

$$\left|\frac{\partial^{m+n} w_1}{\partial x^m \partial y^n}(x,y)\right| \leqslant C(1+\varepsilon^{2-m})\varepsilon^{-n} e^{-\beta y/\varepsilon} \qquad 0 \leqslant m+n \leqslant 3,$$

$$\left|\frac{\partial^{m+n} w_2}{\partial x^m \partial y^n}(x,y)\right| \leqslant C(1+\varepsilon^{2-n})\varepsilon^{-m} e^{-\beta x/\varepsilon} \qquad 0 \leqslant m+n \leqslant 3,$$

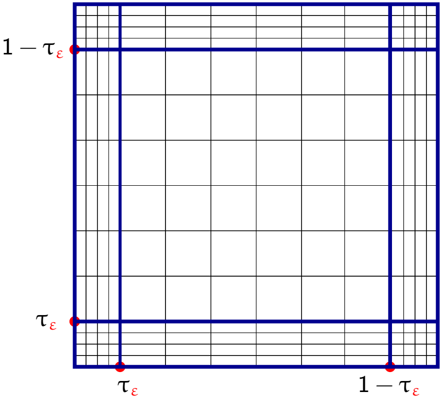
$$\left|\frac{\partial^{m+n} z_1}{\partial x^m \partial y^n}(x,y)\right| \leqslant C\varepsilon^{-m-n} e^{-\beta(x+y)/\varepsilon} \qquad 0 \leqslant m+n \leqslant 3,$$

with analogous bounds for w_3, w_4, z_2, z_3 and z_4.

The Shishkin mesh

We use the same Shishkin mesh as for the Finite Difference method. Define

$$\tau_\varepsilon = \min\left\{\frac{1}{4},\ 2\varepsilon\beta^{-1}\ln N\right\}.$$



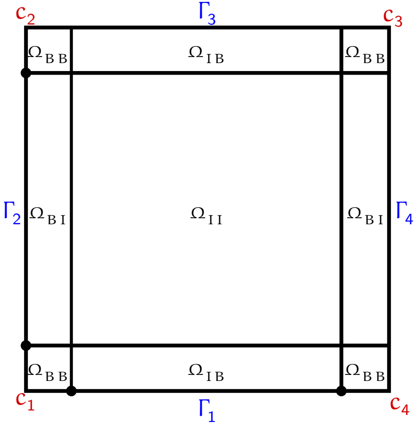
The Shishkin mesh

We will consider the case where ε is so small that

$$\tau_\varepsilon = 2\varepsilon\beta^{-1}\ln N.$$

Partition Ω as follows: $\bar{\Omega} = \Omega_{II} \cup \Omega_{BI} \cup \Omega_{IB} \cup \Omega_{BB}$, where

$$\begin{aligned}
\Omega_{II} &= [\tau_\varepsilon, 1-\tau_\varepsilon] \times [\tau_\varepsilon, 1-\tau_\varepsilon],\\
\Omega_{BI} &= ([0,\tau_\varepsilon] \cup [1-\tau_\varepsilon, 1]) \times [\tau_\varepsilon, 1-\tau_\varepsilon],\\
\Omega_{IB} &= [\tau_\varepsilon, 1-\tau_\varepsilon] \times ([0,\tau_\varepsilon] \cup [1-\tau_\varepsilon, 1]),\\
\Omega_{BB} &= ([0,\tau_\varepsilon] \times ([0,\tau_\varepsilon] \cup [1-\tau_\varepsilon, 1]))\\
&\quad \cup ([1-\tau_\varepsilon, 1] \times ([0,\tau_\varepsilon] \cup [1-\tau_\varepsilon, 1])).
\end{aligned}$$

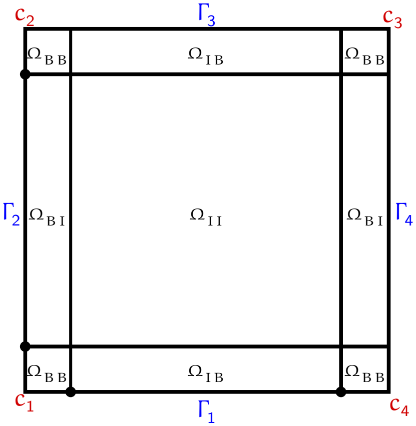


The Shishkin mesh

In the case of interest, $\varepsilon \leqslant N^{-1}$, and so $\tau = 2\varepsilon\beta^{-1}\ln N$. Thus, for any point $(x, y) \in \Omega_{II}$,

$$e^{-\beta x/\varepsilon} \leqslant e^{-\beta\tau/\varepsilon} = N^{-2},$$

$$e^{-\beta y/\varepsilon} \leqslant e^{-\beta\tau/\varepsilon} = N^{-2}.$$



$$\|e^{-\beta(x+y)/\varepsilon}\|_{0,\Omega/\Omega_{BB}} \leqslant \frac{\varepsilon}{\beta}N^{-2};$$

$$\|e^{-\beta(x+y)/\varepsilon}\|_{0,\Omega_{BB}} = \frac{\varepsilon}{2\beta}.$$

$$\|e^{-\beta y/\varepsilon}\|^2_{0,\Omega_{II}\cup\Omega_{BI}} = \|e^{-\beta x/\varepsilon}\|^2_{0,\Omega_{II}\cup\Omega_{IB}} \leqslant \frac{\varepsilon}{2\beta}N^{-4}.$$

$$\|e^{-\beta y/\varepsilon}\|^2_{0,\Omega_{BB}\cup\Omega_{IB}} = \|e^{-\beta x/\varepsilon}\|^2_{0,\Omega_{BB}\cup\Omega_{BI}} \leqslant \frac{\varepsilon}{2\beta}.$$

Interpolation

Given a one-dimensional mesh, Ω_x^N, let V^N be the associated space of piecewise linear functions.

Let $I^N : C[0,1] \to V^N[0,1]$ be the usual piecewise linear Lagrange interpolation operator associated with V^N.

Let $p \in [2,\infty]$ and $\phi \in W^{2,p}[0,1]$. Then the piecewise linear interpolant $I_N\phi$ of ϕ satisfies the bounds

$$\|\phi - I_N\phi\|_{0,p,[x_{i-1},x_i]} + h_i\|(\phi - I_N\phi)'\|_{0,p,[x_{i-1},x_i]} \leqslant C \min\left\{h_i\|\phi'\|_{0,p,[x_{i-1},x_i]},\ h_i^2\|\phi''\|_{0,p,[x_{i-1},x_i]}\right\}.$$

From standard inverse inequalities in one dimension one sees easily that

$$h_x \left\|\frac{\partial\psi}{\partial x}\right\|_{0,K} + k_y \left\|\frac{\partial\psi}{\partial y}\right\|_{0,K} \leqslant \|\psi\|_{0,K} \quad \forall\psi \in V^{N_x,N_y}(\Omega),\ \forall K \in T^{N_x,N_y}(\Omega), \tag{5}$$

where the rectangle K has size $h_x \times k_y$.

Interpolation

The Shishkin mesh is highly anisotropic on $\Omega_{IB} \cup \Omega_{BI}$, and to obtain satisfactory interpolation error estimates on this region one uses the sharp anisotropic interpolation analysis of [Apel, 1999, Apel and Dobrowolski, 1992]:

Lemma

Let τ be any mesh rectangle of size $h_x \times k_y$. Let $\phi \in H^2(\tau)$. Then its piecewise bilinear nodal interpolant ϕ^I satisfies the bounds

$$\|\phi - \phi^I\|_{0,\tau} \leqslant C\left(h_x^2\|\phi_{xx}\|_{0,\tau} + h_x k_y\|\phi_{xy}\|_{0,\tau} + k_y^2\|\phi_{yy}\|_{0,\tau}\right),$$

$$\|(\phi - \phi^I)_x\|_{0,\tau} \leqslant C\left(h_x\|\phi_{xx}\|_{0,\tau} + k_y\|\phi_{xy}\|_{0,\tau}\right),$$

$$\|(\phi - \phi^I)_y\|_{0,\tau} \leqslant C\left(h_x\|\phi_{xy}\|_{0,\tau} + k_y\|\phi_{yy}\|_{0,\tau}\right).$$

The **anisotropic** nature of the bounds is crucial:

Interpolation

Equipped with these results, we would like to prove that

Lemma

There exists a constant C *such that*

$$\|u - I_{N,N}u\|_{0,\Omega} \leqslant CN^{-2}. \tag{6a}$$

and

$$\varepsilon\|\nabla(u - I_{N,N}u)\|_{0,\Omega} \leqslant C(N^{-2} + \varepsilon^{1/2}N^{-1}\ln N). \tag{6b}$$

Here we will give an account of how the bound in (6b) is obtained.

Interpolation

From the solution decomposition,

$$\varepsilon\|\nabla(u - I_{N,N}u)\|_{0,\Omega} = \varepsilon \left\| \nabla \left((I - I_{N,N}) \left(v + \sum_{k=1}^{4} w_k + \sum_{k=1}^{4} z_k \right) \right) \right\|_{0,\Omega}.$$

Each term in this decomposition is bounded separately.

First, standard arguments give,

$$\varepsilon \left\| \frac{\partial}{\partial x}(v - I_{N,N}v) \right\|_{0,\Omega} \leqslant C\varepsilon N^{-1}|v|_{2,\Omega} \leqslant CN^{-2}.$$

Interpolation

Recall that w_1 is the term associated with Γ_1, and, so $w_1(x,y) \sim e^{-y\beta/\varepsilon}$. That fact, and the anisotropic interpolation results, give

$$\begin{aligned}
\varepsilon \left\| \frac{\partial}{\partial x}(w_1 - I_{N,N} w_1) \right\|_{0,\Omega_{II}\cup\Omega_{BI}} &\leqslant C\varepsilon N^{-1} \Bigg(\left\| \frac{\partial^2 w_1}{\partial x^2} \right\|_{0,\Omega_{II}\cup\Omega_{BI}} \\
&\qquad + \left\| \frac{\partial^2 w_1}{\partial x \partial y} \right\|_{0,\Omega_{II}\cup\Omega_{BI}} \Bigg) \\
&\leqslant C\varepsilon N^{-1} \left(1 + \max_{(x,y)\in\Omega_{II}\cup\Omega_{BI}} \varepsilon^{-1} e^{-\beta y/\varepsilon} \right) \\
&\leqslant CN^{-2}.
\end{aligned}$$

Here we have used that $\varepsilon \leqslant N^{-1}$ and that, in this region, $e^{-\beta y/\varepsilon} \approx N^{-2}$.

Interpolation

On $\Omega_{IB} \cup \Omega_{BB}$,

$$\varepsilon \left\| \frac{\partial}{\partial x}(w_1 - I_{N,N} w_1) \right\|_{0,\Omega_{IB}\cup\Omega_{BB}} \leqslant C\varepsilon \Bigg[N^{-1} \left\| \frac{\partial^2 w_1}{\partial x^2} \right\|_{0,\Omega_{IB}\cup\Omega_{BB}} + \varepsilon N^{-1} (\ln N) \left\| \frac{\partial^2 w_1}{\partial x \partial y} \right\|_{0,\Omega_{IB}\cup\Omega_{BB}} \Bigg] \leqslant CN^{-2}.$$

Thus $\varepsilon \left\| \frac{\partial}{\partial x}(w_1 - I_{N,N} w_1) \right\|_{0,\Omega} \leqslant CN^{-2}$.

Interpolation

Next recall that w_2 is the component associated with the edge layer near Γ_2. So, roughly, $w_2(x, y) \sim e^{-x\beta/\varepsilon}$.

Similar to above, we can show that

$$\varepsilon \|(w_2 - I_{N,N} w_2)_x\|_{0,\Omega_{II} \cup \Omega_{IB}} \leqslant N^{-2}.$$

However, the most significant term is

$$\varepsilon \left\| \frac{\partial}{\partial x}(w_2 - I_{N,N} w_2) \right\|_{0,\Omega_{BI} \cup \Omega_{BB}} \leqslant$$

$$C\varepsilon \left[\varepsilon N^{-1} (\ln N) \left\| \frac{\partial^2 w_2}{\partial x^2} \right\|_{0,\Omega_{BI} \cup \Omega_{BB}} + N^{-1} \left\| \frac{\partial^2 w_2}{\partial x \partial y} \right\|_{0,\Omega_{BI} \cup \Omega_{BB}} \right]$$

$$\leqslant C\varepsilon^{1/2} N^{-1} \ln N.$$

Consequently,

$$\varepsilon \left\| \frac{\partial}{\partial x}(w_2 - I_{N,N} w_2) \right\|_{0,\Omega} \leqslant C(N^{-2} + \varepsilon^{1/2} N^{-1} \ln N).$$

Interpolation

Analogous results are valid for w_3, w_4 and the corner layer terms, z_1, z_2, z_3, z_4.

Gathering these results yields $\varepsilon \left\| \frac{\partial}{\partial x}(u - I_{N,N}u) \right\|_{0,\Omega} \leqslant C(N^{-2} + \varepsilon^{1/2} N^{-1} \ln N)$. The same estimate is valid for $\varepsilon \| \frac{\partial}{\partial y}(u - I_{N,N}u) \|_{0,\Omega}$.

It is then clear that

Theorem

There exists a constant C such that

$$\|u - I_{N,N}u\|_{0,\Omega} + \varepsilon \|\nabla(u - I_{N,N}u)\|_{0,\Omega} \leqslant C(N^{-2} + \varepsilon^{1/2} N^{-1} \ln N).$$

The Galerkin FEM

The variational formulation of (4) is: find $u \in H_0^1(\Omega)$ such that

$$B(u, v) := \varepsilon^2(\nabla u, \nabla v) + (bu, v) = (f, v) \quad \forall v \in H_0^1(\Omega).$$

Define an associated *energy norm*

$$\|v\|_{\varepsilon} := \left\{\varepsilon^2\|\nabla v\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2\right\}^{1/2}.$$

This bilinear form is coercive with respect to this norm:

$$B(v, v) = \varepsilon^2 \left\|\frac{\partial v}{\partial x}\right\|_{0,\Omega}^2 + \varepsilon^2 \left\|\frac{\partial v}{\partial y}\right\|_{0,\Omega}^2 + b\|v\|_{0,\Omega}^2 \geqslant \min\{1, 2\beta^2\}\,\|v\|_{\varepsilon}^2 \quad \forall\, v \in H_0^1(\Omega)$$

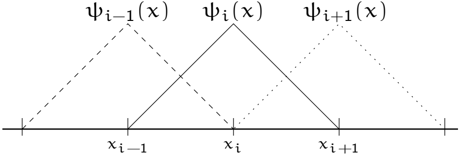
Furthermore it is continuous

$$|B(v, w)| \leqslant (2 + \|b\|_{0,\infty,\Omega})\|v\|_{\varepsilon}\|w\|_{\varepsilon} \quad \forall\, v, w \in H_0^1(\Omega).$$

Standard Bilinear FEs

Our finite element space is the space of piecewise bilinear functions on a tensor product mesh with N intervals in each coordinate direction. We denote this $V_{N,N}$.

- Form a one dimensional mesh, ω^N.
- Let ψ_i^N be the usual "hat" function associated with node i.

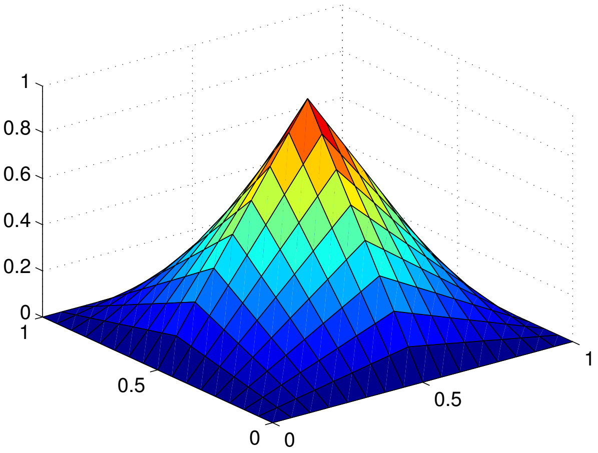


- $V_{N,N}$ is the space which has as a basis

$$\left\{ \psi_i^N(x)\psi_j^N(y) \right\}_{j=1:N-1}^{i=1:N-1}$$

Standard Bilinear FEs

Each basis function in this space resembles that shown below (for a reference element).



Standard Bilinear FEs

Define the Galerkin finite element approximation $u_{N,N} \in V_0^{N,N}(\Omega)$

$$B(u_{N,N}, v_{N,N}) = (f, v_{N,N}) \quad \forall v_{N,N} \in V_0^{N,N}(\Omega).$$

Since $u_{N,N}$ has $(N-1)^2$ degrees of freedom, we need $(N-1)^2$ equations to solve them. These equations are obtained by taking each basis function in turn as **test** function.

Standard Bilinear FEs

Classical finite element arguments based on coercivity and Galerkin orthogonality yields the quasi-optimal bound

$$\|u - u_{N,N}\|_{\varepsilon} \leqslant C \inf_{\phi \in V_0^{N,N}(\Omega)} \|u - \phi\|_{\varepsilon} \leqslant \|u - I_{N,N}u\|_{\varepsilon}.$$

It then follows that...

Theorem

There exists a constant C such that

$$\|u - u_{N,N}\|_{\varepsilon} \leqslant C(N^{-2} + \varepsilon^{1/2} N^{-1} \ln N).$$

Numerical Example

Example

$$-\varepsilon^2 \Delta u + \left(1 + x^2 y^2 e^{xy/2}\right) u = f \quad \text{on } \Omega := (0,1)^2,$$

where f and the boundary conditions are chosen so that

$$u = x^3(1+y^2) + \sin(\pi x^2) + \cos(\pi y/2) + (x+y)\left(e^{-2x/\varepsilon} + e^{-2(1-x)/\varepsilon} + e^{-3y/\varepsilon} + e^{-3(1-y)/\varepsilon}\right).$$

ε^2	$N = 2^4$	$N = 2^6$	$N = 2^8$	$N = 2^{10}$
1	3.395e-1	8.714e-2	2.190e-2	5.482e-3
10^{-2}	4.618e-1	1.572e-1	4.214e-2	1.070e-2
10^{-4}	2.287e-1	1.578e-1	7.228e-2	2.510e-2
10^{-6}	7.220e-2	4.979e-2	2.280e-2	7.921e-3
10^{-8}	2.361e-2	1.574e-2	7.211e-3	2.504e-3
10^{-10}	9.621e-3	4.992e-3	2.280e-3	7.919e-4
10^{-12}	6.787e-3	1.619e-3	7.214e-4	2.504e-4
10^{-14}	6.435e-3	6.265e-4	2.292e-4	7.920e-5

Other norms

The above results are somewhat suspect looking... although the method does resolve layers, the error, in both theory and practice, shows an ε-dependency.

However, it is observed that (subject to sufficient regularity),

$$\|u - u_{N,N}\|_{\infty,\bar{\Omega}^N} \leqslant CN^{-2}.$$

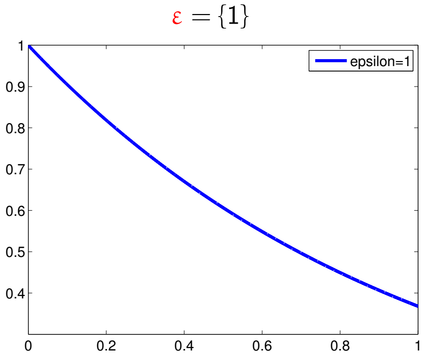
So, in some sense, the difficulty is with the norm, rather than the method.

Consider this very simple one-dimensional singularly perturbed reaction-diffusion problem:

$$-\varepsilon^2 u''(x) + u(x) = 0 \text{ on } (0,1),$$

$$u(0) = 1, u(1) = e^{-1/\epsilon} (\approx 0).$$

Its solution is $u(x) = e^{-x/\epsilon}$.



$$\|u\|_\infty := \max_{0 \leqslant x} |u(x)| = 1, \quad \text{but} \quad \|u\|_0 := \sqrt{\int_0^1 (u(x))^2 dx} \approx \sqrt{\varepsilon}.$$

As $\varepsilon \to 0$, we get that $\|u\|_0 \to 0$, even though $\|u\|_\infty \to 1$.

Trivially, this shows that $u^h \equiv 0$ is a terrible approximation to u with

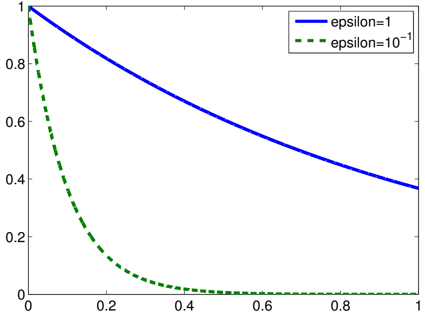
Consider this very simple one-dimensional singularly perturbed reaction-diffusion problem:

$$-\varepsilon^2 u''(x) + u(x) = 0 \text{ on } (0,1),$$

$$u(0) = 1, u(1) = e^{-1/\epsilon} (\approx 0).$$

Its solution is $u(x) = e^{-x/\epsilon}$.

$\varepsilon = \{1, 10^{-1}\}$



$$\|u\|_\infty := \max_{0 \leqslant x} |u(x)| = 1, \quad \text{but} \quad \|u\|_0 := \sqrt{\int_0^1 (u(x))^2 dx} \approx \sqrt{\varepsilon}.$$

As $\varepsilon \to 0$, we get that $\|u\|_0 \to 0$, even though $\|u\|_\infty \to 1$.

Trivially, this shows that $u^h \equiv 0$ is a terrible approximation to u with respect to $\|\cdot\|_\infty$, but rather good with respect to $\|\cdot\|_0$.

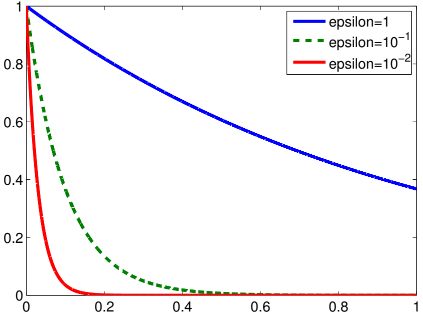
Consider this very simple one-dimensional singularly perturbed reaction-diffusion problem:

$$-\varepsilon^2 u''(x) + u(x) = 0 \text{ on } (0,1),$$

$$u(0) = 1, u(1) = e^{-1/\epsilon} (\approx 0).$$

Its solution is $u(x) = e^{-x/\epsilon}$.

$$\varepsilon = \{1, 10^{-1}, 10^{-2}\}$$



$$\|u\|_\infty := \max_{0 \leqslant x} |u(x)| = 1, \quad \text{but} \quad \|u\|_0 := \sqrt{\int_0^1 (u(x))^2 dx} \approx \sqrt{\varepsilon}.$$

As $\varepsilon \to 0$, we get that $\|u\|_0 \to 0$, even though $\|u\|_\infty \to 1$.

Trivially, this shows that $u^h \equiv 0$ is a terrible approximation to u with respect to $\|\cdot\|_\infty$, but rather good with respect to $\|\cdot\|_0$.

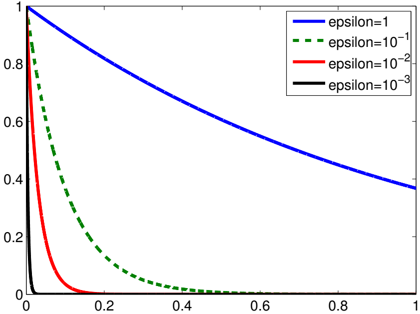
Consider this very simple one-dimensional singularly perturbed reaction-diffusion problem:

$$-\varepsilon^2 u''(x) + u(x) = 0 \text{ on } (0,1),$$

$$u(0) = 1, u(1) = e^{-1/\epsilon} (\approx 0).$$

Its solution is $u(x) = e^{-x/\epsilon}$.

$\varepsilon = \{1, 10^{-1}, 10^{-2}, 10^{-3}\}$



$$\|u\|_\infty := \max_{0 \leqslant x} |u(x)| = 1, \quad \text{but} \quad \|u\|_0 := \sqrt{\int_0^1 (u(x))^2 dx} \approx \sqrt{\varepsilon}.$$

As $\varepsilon \to 0$, we get that $\|u\|_0 \to 0$, even though $\|u\|_\infty \to 1$.

Trivially, this shows that $u^h \equiv 0$ is a terrible approximation to u with respect to $\|\cdot\|_\infty$, but rather good with respect to $\|\cdot\|_0$.

Slightly less trivially, try solving this problem with a standard Galerkin FEM. The weak form is:

$$B(u, v) := \int_0^1 \varepsilon^2 u'(x)v'(x) + u(x)v(x), \qquad (f, v) := \int_0^1 f(x)v(x),$$

and find $u \in H_0^1(0, 1)$.

$$B(u, v) = (f, v) \quad \text{for all } v \in H_0^1(0, 1).$$

The energy norm is

$$\|g\|_{\varepsilon} := \left(\varepsilon^2 \|g'\|_0^2 + \|g\|_0^2\right)^{1/2}.$$

But this norm is weak, since

$$\left(\varepsilon^2 \|u'\|_0 + \|u\|_0\right)^{1/2} \approx \sqrt{\varepsilon}.$$

In contrast,

$$\left(\varepsilon \|u'\|_2 + \|u\|_2\right)^{1/2} \approx 1.$$

Slightly less trivially, try solving this problem with a standard Galerkin FEM. The weak form is:

$$B(u,v) := \int_0^1 \varepsilon^2 u'(x)v'(x) + u(x)v(x), \qquad (f,v) := \int_0^1 f(x)v(x),$$

and find $u \in H_0^1(0,1)$.

$$B(u,v) = (f,v) \quad \text{for all } v \in H_0^1(0,1).$$

The energy norm is

$$\|g\|_\varepsilon := \left(\varepsilon^2 \|g'\|_0^2 + \|g\|_0^2\right)^{1/2}.$$

But this norm is weak, since

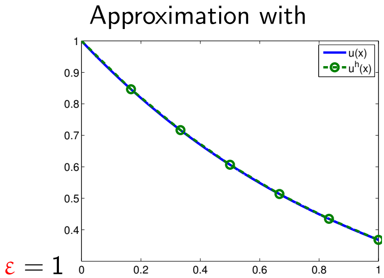
$$\left(\varepsilon^2 \|u'\|_0 + \|u\|_0\right)^{1/2} \approx \sqrt{\varepsilon}.$$

In contrast,

$$\left(\varepsilon \|u'\|_2 + \|u\|_2\right)^{1/2} \approx 1.$$

Suppose we did try to solve our simple ODE with a Galerkin FEM with linear elements on a uniform mesh... Clearly, even though this is a "good" estimate at mesh points, it is clear that

$$\|u - u_N\|_{\infty,\Omega} \sim \mathcal{O}(1).$$



It is known ([Bagaev and Shaĭdurov, 1998], [Farrell et al., 2000]) that

$$\|u - u^N\|_{\varepsilon} \leqslant CN^{-1/2}.$$

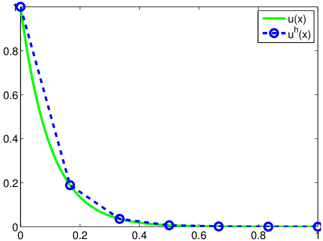
So we now have two problems with the energy norm:

- it appears to show robust convergence even when layers are not being resolved.
- On the layer-resolving Shishkin mesh, the $\mathcal{O}(\varepsilon^{1/2}N^{-1}\ln N)$ quantity demonstrates that this norm is not "balanced".

Suppose we did try to solve our simple ODE with a Galerkin FEM with linear elements on a uniform mesh... Clearly, even though this is a "good" estimate at mesh points, it is clear that

$$\|u - u_N\|_{\infty,\Omega} \sim \mathcal{O}(1).$$

Approximation with $\varepsilon = 10^{-1}$



It is known ([Bagaev and Shaĭdurov, 1998], [Farrell et al., 2000]) that

$$\|u - u^N\|_{\varepsilon} \leqslant CN^{-1/2}.$$

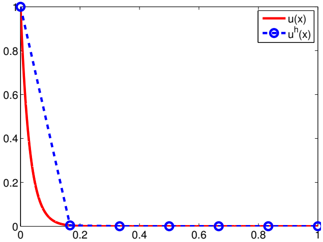
So we now have two problems with the energy norm:

- it appears to show robust convergence even when layers are not being resolved.
- On the layer-resolving Shishkin mesh, the $\mathcal{O}(\varepsilon^{1/2}N^{-1}\ln N)$ quantity demonstrates that this norm is not "balanced".

Suppose we did try to solve our simple ODE with a Galerkin FEM with linear elements on a uniform mesh... Clearly, even though this is a "good" estimate at mesh points, it is clear that

$$\|u - u_N\|_{\infty,\Omega} \sim \mathcal{O}(1).$$

Approximation with $\varepsilon = 10^{-2}$



It is known ([Bagaev and Shaĭdurov, 1998], [Farrell et al., 2000]) that

$$\|u - u^N\|_{\varepsilon} \leqslant CN^{-1/2}.$$

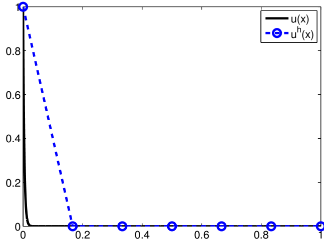
So we now have two problems with the energy norm:

- it appears to show robust convergence even when layers are not being resolved.
- On the layer-resolving Shishkin mesh, the $\mathcal{O}(\varepsilon^{1/2}N^{-1}\ln N)$ quantity demonstrates that this norm is not "balanced".

Suppose we did try to solve our simple ODE with a Galerkin FEM with linear elements on a uniform mesh... Clearly, even though this is a "good" estimate at mesh points, it is clear that

$$\|u - u_N\|_{\infty,\Omega} \sim \mathcal{O}(1).$$

Approximation with $\varepsilon = 10^{-3}$



It is known ([Bagaev and Shaĭdurov, 1998], [Farrell et al., 2000]) that

$$\|u - u^N\|_{\varepsilon} \leqslant CN^{-1/2}.$$

So we now have two problems with the energy norm:

- it appears to show robust convergence even when layers are not being resolved.
- On the layer-resolving Shishkin mesh, the $\mathcal{O}(\varepsilon^{1/2}N^{-1}\ln N)$ quantity demonstrates that this norm is not "balanced".

There are several approaches to resolving the problem of the weakness of the usual energy norm for this problem:

(a) Analyse a standard FEM (on a suitable mesh), but with respect to a stronger norm, such as

$$\|v\|_{\mathrm{bal}} := \left(\varepsilon\|\nabla v\|_0^2 + \|v\|_0^2\right)^{1/2}.$$

This is done in [Roos and Schopf, 2014], and also [Melenk and Xenophontos, 2015].

(b) Design a new FEM for which the natural induced norm *is* balanced. E.g.,
   - In [Lin and Stynes, 2012], this is done using a first-order system approach.
   - In FOSLS-type setting, see [Adler et al., 2016]
   - In [Roos and Schopf, 2014], a C^0 interior penalty (CIP) method is constructed.

A three dimensional problem

A singularly perturbed problem in 3D

Solve the following reaction-diffusion equation posed on the unit cube:

$$-\varepsilon^2(u_{xx} + u_{yy} + u_{zz}) + b(x, y, z)u(x, y, z) = f(x, y, z)$$

Example (Naresh Chadha and Natalia Kopteva)

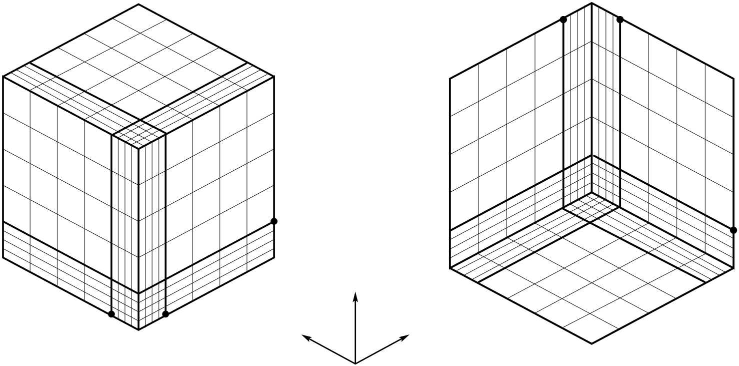
In (**??**), set $b \equiv 1$ and f such that

$$u = \left(\cos\left(\frac{\pi x}{2}\right) - \frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right)\left(1 - y - \frac{e^{-y/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right)$$

$$\left(1 - z^2 - \frac{e^{-z/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right).$$

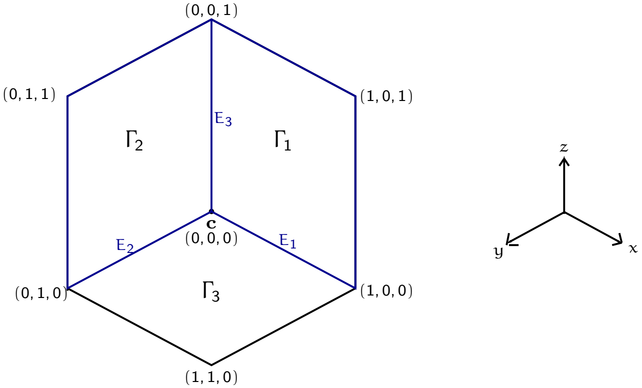
This problem exhibits 1D exponential layers near the faces of the domain, $(0, y, z)$, $(x, 0, z)$ and $(x, y, 0)$, as well as 2D layers near the edges, $(0, 0, z)$, $(0, y, 0)$ and $(x, 0, 0)$, and a 3D layer at the origin $(0, 0, 0)$.

A three dimensional problem



The above problem is artificially simplified. In general, solutions to 3D problems feature six 1D, twelve 2D and eight 3D layers. Therefore, when the interior of the domain is included, there are 27 distinct regions to be analysed. However, it captures the essence of the problem: it features 1D, 2D and 3D layers.

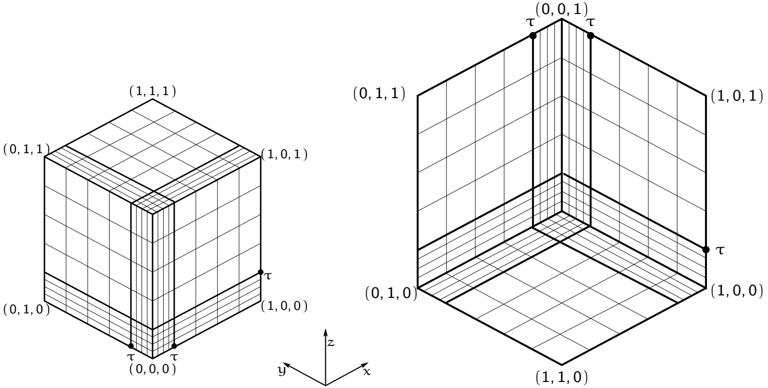
A three dimensional problem



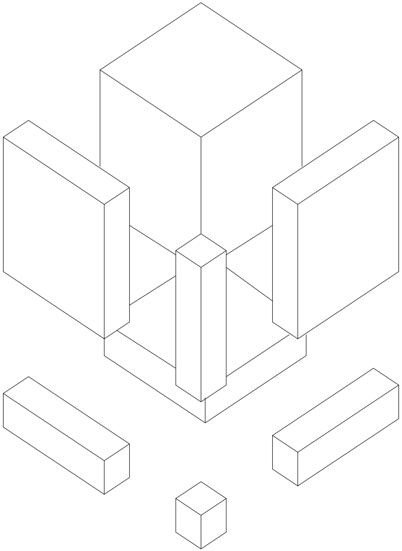
A three dimensional problem

Let τ be the transition parameter that specifies the point where the mesh transitions between coarse and fine, defined as

$$\tau = \min\left\{\frac{1}{2}, \frac{2\varepsilon \ln N}{\beta}\right\}.$$



A three dimensional problem



The decomposition is a variant of that in [Shishkin and Shishkina, 2009, §3.2]. It gives u as the sum of

- a regular component v,
- components r_1, r_2 and r_3, corresponding to the 1D layers associated with Γ_i, $i = 1, 2, 3$,
- components s_1, s_2 and s_3, corresponding to the 2D layers associated with E_i, $i = 1, 2, 3$, and
- a component t, corresponding to the 3D layer associated with the corner c.

Lemma (Theorem 3.2.2)

Let $b, f \in \mathcal{C}^{4,\alpha}(\bar{\Omega})$, $\alpha \in (0,1)$. *Then* u *can be decomposed as*

$$u = v + \sum_{i=1}^{3} r_i + \sum_{i=1}^{3} s_i + t, \tag{7}$$

where for $l, m, n \geqslant 0$ *there exists a constant* C*, such that*

$$\left| \frac{\partial^{l+m+n} v}{\partial x^l \partial y^m \partial z^n}(x,y,z) \right| \leqslant C\left(1 + \varepsilon^{2-l-m-n}\right), \tag{8a}$$

$$\left| \frac{\partial^{l+m+n} r_1}{\partial x^l \partial y^m \partial z^n}(x,y,z) \right| \leqslant C(1 + \varepsilon^{2-l-n})\varepsilon^{-m} e^{-\beta y/\varepsilon}, \tag{8b}$$

$$\left| \frac{\partial^{l+m+n} s_1}{\partial x^l \partial y^m \partial z^n}(x,y,z) \right| \leqslant C(1 + \varepsilon^{2-l})\varepsilon^{-m-n} e^{-\beta(y+z)/\varepsilon}, \tag{8c}$$

$$\left| \frac{\partial^{l+m+n} t}{\partial x^l \partial y^m \partial z^n}(x,y,z) \right| \leqslant C\varepsilon^{-l-m-n} e^{-\beta(x+y+z)/\varepsilon}, \tag{8d}$$

The following lemma provides bounds on derivatives in the L^2-norm required for the analysis of the interpolation error.

Lemma (Partial statement)

For $0 \leqslant l + m + n \leqslant 3$, *there exists a constant*, C, *such that*

$$\left\| \frac{\partial^{l+m+n} r_1}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega_{UBU}} \leqslant C(1 + \varepsilon^{2-l-n})\varepsilon^{1/2-m},$$

$$\left\| \frac{\partial^{l+m+n} r_1}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega \setminus \Omega_{UBU}} \leqslant C(1 + \varepsilon^{2-l-n})\varepsilon^{1/2-m} N^{-2},$$

...

$$\left\| \frac{\partial^{l+m+n} t}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega_{BBB}} \leqslant C\varepsilon^{3/2-l-m-n},$$

$$\left\| \frac{\partial^{l+m+n} t}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega \setminus \Omega_{BBB}} \leqslant C\varepsilon^{3/2-l-m-n} N^{-2}.$$

The analysis proceeds much like the 2D case, but is more intricate.

The main ingredients are

(i) Construct a trilinear interpolation operator, $I_{N,N,N}$;

(ii) Apply anisotropic interpolation estimates on each brick;

(iii) Use these to prove that $\|u - I_{N,N,N}u\|_{0,\Omega} \leqslant CN^{-2}$. and that $\varepsilon\|\nabla(u - I_{N,N,N}u)\|_{0,\Omega} \leqslant C\varepsilon^{1/2}N^{-1}\ln N$.

(iv) Define the FE space $V_{N,N,N}(\Omega) = \text{span}\left\{\psi_i^N(x)\psi_j^N(y)\psi_k^N(z)\right\}$.

(v) Define Galerkin FEM as: *find* $u_{N,N,N} \in V_{N,N,N}(\Omega)$ *such that*

$$B(u_{N,N,N}, v_{N,N,N}) = (f, v_{N,N,N}) \quad \forall v_{N,N,N} \in V_{N,N,N}(\Omega).$$

Theorem

Then there exists a constant C, *independent of* ε *and* N, *such that*

$$\|u - u_{N,N,N}\|_\varepsilon \leqslant C(N^{-2} + \varepsilon^{1/2}N^{-1}\ln N).$$

Numerical results support the theory.

ε^2	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
1	4.201e-3	2.097e-3	1.048e-3	5.241e-4	2.620e-4	1.310e-4
10^{-2}	4.264e-2	2.196e-2	1.107e-2	5.547e-3	2.775e-3	1.388e-3
10^{-4}	2.141e-2	1.477e-2	9.417e-3	5.700e-3	3.336e-3	1.909e-3
10^{-6}	7.060e-3	4.815e-3	3.069e-3	1.858e-3	1.087e-3	6.221e-4
10^{-8}	2.503e-3	1.540e-3	9.740e-4	5.893e-4	3.449e-04	1.973e-4
10^{-10}	1.370e-3	5.245e-4	3.100e-4	1.865e-4	1.091e-4	6.242e-5
10^{-12}	1.200e-3	2.557e-4	1.038e-4	5.928e-5	3.452e-5	1.974e-5

This shows that the theory is sharp. When $\varepsilon = 1$, it is obvious that the error is proportional to N^{-1}. For small N and ε, the $\mathcal{O}(N^{-2})$ term in the error appears to dominate. For larger N, it is clear that the $\mathcal{O}(\varepsilon^{1/2} N^{-1} \ln N)$ term is dominating. These results can be further visualised in the log-log plot below.

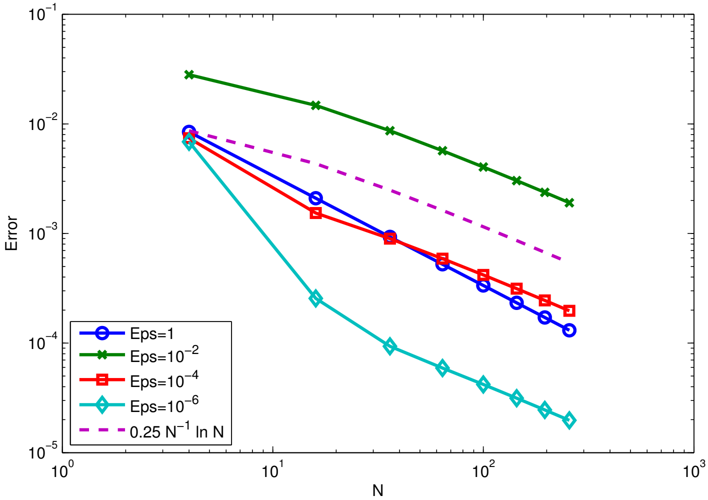


Figure: A log-log plot of the errors for the standard Galerkin FEM in three dimensions for various N and ε.

References

Adler, J., MacLachlan, S., and Madden, N. (2016).
A first-order system Petrov-Galerkin discretization for a reaction-diffusion problem on a fitted mesh.
IMA Journal of Numerical Analysis, 36(3):1281–1309.
10.1093/imanum/drv045.

Apel, T. (1999).
Anisotropic finite elements: Local estimates and applications.
Advances in Numerical Mathematics. B.G. Teubner, Stuttgart.

Apel, T. and Dobrowolski, M. (1992).
Anisotropic interpolation with applications to the finite element method.
Computing, 47:277–293.

Bagaev, B. M. and Shaĭdurov, V. V. (1998).
Setochnye metody resheniya zadach s pogranichnym sloem. Chast 1.
"Nauka", Sibirskoe Predpriyatie RAN, Novosibirsk.

References

Clavero, C., Gracia, J. L., and O'Riordan, E. (2005).
A parameter robust numerical method for a two dimensional reaction-diffusion problem.
Math. Comp., 74(252):1743–1758.

Farrell, P. A., Hegarty, A. F., Miller, J. J. H., O'Riordan, E., and Shishkin, G. I. (2000).
Robust Computational Techniques for Boundary Layers.
Number 16 in Applied Mathematics. Chapman & Hall/CRC, Boca Raton, U.S.A.

Lin, R. and Stynes, M. (2012).
A balanced finite element method for singularly perturbed reaction-diffusion problems.
SIAM J. Numer. Anal., 50(5):2729–2743.

References

Linß, T. (2010).
Layer-adapted meshes for reaction-convection-diffusion problems, volume 1985 of *Lecture Notes in Mathematics*.
Springer-Verlag, Berlin.

Linß, T. and Madden, N. (2004).
A finite element analysis of a coupled system of singularly perturbed reaction-diffusion equations.
Appl. Math. Comp., 148:869–880.

Liu, F., Madden, N., Stynes, M., and Zhou, A. (2009).
A two-scale sparse grid method for a singularly perturbed reaction–diffusion problem in two dimensions.
IMA J. Numer. Anal., 29(4):986–1007.

Melenk, J. and Xenophontos, C. (2015).
Robust exponential convergence of hp-fem in balanced norms for singularly perturbed reaction-diffusion equations.
Calcolo, pages 1–28.

References

Roos, H.-G. and Schopf, M. (2014).
Convergence and stability in balanced norms of finite element methods on Shishkin meshes for reaction-diffusion problems.
ZAMM, Z. Angew. Math. Mech.
doi: 10.1002/zamm.201300226.

Roos, H.-G., Stynes, M., and Tobiska, L. (2008).
Robust Numerical Methods for Singularly Perturbed Differential Equations, volume 24 of *Springer Series in Computational Mathematics*.
Springer-Verlag, Berlin, 2nd edition.

Russell, S. and Madden, N. (2017a).
Analysis of a Galerkin finite element method applied to a singularly perturbed reaction-diffusion problem in three dimensions.
(under review).

References

Russell, S. and Madden, N. (2017b).
An introduction to the analysis and implementation of sparse grid finite element methods.
Computational Methods in Applied Mathematics, 17(2):299–322.

Shishkin, G. I. and Shishkina, L. P. (2009).
Difference methods for singular perturbation problems, volume 140 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*.
CRC Press, Boca Raton, FL.